Low-energy asymptotic expansion of the Green function for one-dimensional Fokker–Planck

and Schrödinger equations

Low-energy asymptotic expansion of the Green function for one-dimensional Fokker–Planck and Schrödinger equations

Toru Miyazawa

Department of Physics, Gakushuin University, Tokyo 171-8588, Japan

E-mail: toru.miyazawa@gakushuin.ac.jp

Abstract
We consider Schrödinger equations and Fokker–Planck equations in one
dimension, and study the low-energy asymptotic behavior of the Green function
using a new method. In this method, the coefficient of the expansion in powers
of the wave number can be systematically calculated to arbitrary order, and the
behavior of the remainder term can be analyzed on the basis of an expression in
terms of transmission and reflection coefficients. This method is applicable to
a wide variety of potentials which may not necessarily be finite as $x \to \pm\infty$.

PACS numbers: 03.65.Nk, 02.30.Hq, 02.50.Ey

1. Introduction

We consider the one-dimensional Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V_S(x)\psi(x) = k^2\psi(x), \tag{1.1}$$

or the equivalent Fokker–Planck equation [1]

$$-\frac{d^2}{dx^2}\phi(x) + 2\frac{d}{dx}[f(x)\phi(x)] = k^2\phi(x). \tag{1.2}$$

The Fokker–Planck equation (1.2) describes the diffusion process in an external potential
$V(x)$, which is related to the function $f(x)$ in (1.2) by

$$f(x) = -\frac{1}{2}\frac{d}{dx}V(x), \tag{1.3}$$

or $V(x) = -2\int f(x)\,dx$. The correspondence between equations (1.1) and (1.2) is given by
the relations $\phi(x) = e^{-V(x)/2}\psi(x)$ and

$$V_S(x) = f^2(x) + f'(x). \tag{1.4}$$

Here we study the Green function for (1.1) or (1.2). Our analysis is based on the expression of the Green function in terms of reflection coefficients which were derived in [2], and the method of asymptotic expansion for the reflection coefficients presented in [3]. In this method, the high- and low-energy expansions can be treated on an equal footing. The high-energy expansion of the Green function was discussed in a previous paper [4]. In the present paper, we study the low-energy expansion.

The low-energy asymptotic behavior of the Green function and related functions has been studied for many years, and various methods have been proposed [5–15]. (Most of these methods are limited to the cases where $\int_{-\infty}^{\infty} |V_S(x)|\,\mathrm{d}x < \infty$, although there are some specific methods for other types of V_S.) In this paper, we use a method different from any of these previous works, and derive simple results for the expansion in powers of the wave number k to arbitrary order. There have been previously similar attempts at systematically calculating the expansion of the Green function to arbitrary order [9], but the formulae derived in this paper are new, and applicable to a larger class of potentials (including the cases where $V_S(x)$ is not finite as $x \to \pm\infty$). These results are not only of theoretical interest, but also useful for practical calculations.

We assume that the Fokker–Planck potential $V(x)$ is a real-valued function which is piecewise continuously differentiable[1]. We allow $V(x)$ to be either $+\infty$, $-\infty$ or finite as $x \to +\infty$, and similarly for $x \to -\infty$, and we require that $V(x)$ behave steadily and smoothly at spatial infinity. Specifically, we assume that $V(x)$, $f(x)$ and $f'(x)$ are monotone for sufficiently large $|x|$, at both $x \to +\infty$ and $x \to -\infty$. We do not consider the cases where $V(x)$ shows oscillatory or other indeterminate behavior as $x \to \pm\infty$. Note that $V_S(\pm\infty)$ are either finite or $+\infty$. The cases $V_S(\pm\infty) = -\infty$ are not considered here, since such V_S does not correspond to an appropriate Fokker–Planck potential.

Our aim in this paper is to derive an expansion of the Green function in powers of k for $\mathrm{Im}\,k \geqslant 0$. For such an expansion to be possible, it is necessary that $V(x)$ either converge sufficiently rapidly or diverge sufficiently rapidly as $|x| \to \pm\infty$. Let us introduce the following classes of real-valued functions:

$$F_n^{(-)} = \left\{ g \left| \int_{-\infty}^{a} (1 + |x|^n)|g(x)|\,\mathrm{d}x < \infty \text{ for any finite } a \right. \right\}, \qquad (1.5a)$$

$$F_n^{(+)} = \left\{ g \left| \int_{a}^{\infty} (1 + |x|^n)|g(x)|\,\mathrm{d}x < \infty \text{ for any finite } a \right. \right\}, \qquad (1.5b)$$

where n is a nonnegative integer. We derive the low-energy expansion under the condition that

$$V - V_1 \in F_n^{(-)} \quad \text{or} \quad \mathrm{e}^{-V} \in F_n^{(-)} \quad \text{or} \quad \mathrm{e}^{V} \in F_n^{(-)} \qquad (1.6)$$

with some n and some finite constant V_1. (Here $V - V_1$, e^{-V} and e^{V} mean $V(x) - V_1$, $\mathrm{e}^{-V(x)}$ and $\mathrm{e}^{V(x)}$ as functions of x.) This is essentially a condition on the behavior of $V(x)$ as $x \to -\infty$, and the three cases in (1.6) correspond to the cases $V(-\infty) = V_1$, $V(-\infty) = +\infty$ and $V(-\infty) = -\infty$ respectively. Similarly, concerning the behavior of $V(x)$ as $x \to +\infty$, it is required that

$$V - V_2 \in F_m^{(+)} \quad \text{or} \quad \mathrm{e}^{-V} \in F_m^{(+)} \quad \text{or} \quad \mathrm{e}^{V} \in F_m^{(+)} \qquad (1.7)$$

---

[1] The Fokker–Planck equation (1.2) is well defined even when $V(x)$ has a jump discontinuity and $f(x)$ has a delta function [1], although the corresponding $V_S(x)$ does not make sense in such a case. A delta function in $f(x)$ is interpreted as jump conditions for $\phi(x)$ and $\phi'(x)$, in much the same way that a delta function in $V_S(x)$ is interpreted as a jump condition for $\psi'(x)$. The jump conditions require that $\mathrm{e}^{V(x)}\phi(x)$ and $\phi'(x) - 2f(x)\phi(x)$ be continuous. For example, if $f(x) = c\delta(x - x_0)$, then $\phi(x_0 + 0) = \mathrm{e}^{2c}\phi(x_0 - 0)$ and $\phi'(x_0 + 0) = \phi'(x_0 - 0)$. If $f(x)$ has a jump, then $V_S(x)$ has a delta function, and $\psi'(x)$ has a jump. The Green function studied in this paper is meaningful for the Fokker–Planck equation even when V_S does not make sense.

with some m and some finite constant V_2. The three cases in (1.7) correspond to the cases $V(+\infty) = V_2$, $V(+\infty) = +\infty$ and $V(+\infty) = -\infty$. Since there are three cases for $x \to -\infty$ and three cases for $x \to +\infty$, there are nine cases in all. But it is sufficient to consider only the following six cases:

(i) $V(-\infty) = V_1$, $V(+\infty) = V_2$,   (ii) $V(-\infty) = V_1$, $V(+\infty) = +\infty$,
(iii) $V(-\infty) = V_1$, $V(+\infty) = -\infty$,   (iv) $V(-\infty) = +\infty$, $V(+\infty) = +\infty$,
(v) $V(-\infty) = +\infty$, $V(+\infty) = -\infty$,   (vi) $V(-\infty) = -\infty$, $V(+\infty) = -\infty$.

We derive an expansion of the Green function for each of the six cases. (The detailed conditions for the validity of the expansion to order k^N are given by (5.29) and (6.3)).

In section 2 we review the expression of the Green function derived in [2], and make a comment on its application to the Schrödinger equation. In section 3, we explain the method of [3] with explicit calculations. Using this, we derive the expressions for the expansion of the Green function in sections 4 and 5. The remainder term is studied in section 6. In section 7, we discuss the special case where $V_S(x)$ tends to 0 at both $x \to +\infty$ and $x \to -\infty$. Examples of calculations are given in section 8.

2. Reflection coefficients and the Green function

Let $G_S(x, y; k)$ denote the Green function[2] for the Schrödinger equation (1.1), satisfying

$$\left[\frac{\partial^2}{\partial x^2} - V_S(x) + k^2 \right] G_S(x, y; k) = \delta(x - y) \tag{2.1}$$

with the boundary conditions $G_S(x, y; k) \to 0$ as $|x - y| \to \infty$ for $\mathrm{Im}\, k > 0$. For $\mathrm{Im}\, k = 0$, we define[3] $G_S(x, y; k) \equiv \lim_{\epsilon \downarrow 0} G_S(x, y; k + i\epsilon)$. Since $G_S(x, y; k) = G_S(y, x; k)$, without loss of generality we assume $x \geqslant y$.

In this paper, we use the expression of the Green function in terms of reflection coefficients for semi-infinite intervals. First, let us define the transmission and reflection coefficients for finite intervals. For arbitrary a and b ($a \leqslant b$), we define

$$\bar{V}(x) \equiv \begin{cases} V(a) & (x < a) \\ V(x) & (a \leqslant x \leqslant b) \\ V(b) & (b < x), \end{cases} \qquad \bar{f}(x) \equiv -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} \bar{V}(x), \tag{2.2}$$

and consider the Fokker–Planck equation (1.2) with $f(x)$ replaced by $\bar{f}(x)$. Since $\bar{f}(x) = 0$ for $x < a$ and $x > b$, this equation has two solutions of the forms

$$\phi_1(x) = \begin{cases} \tau\, \mathrm{e}^{-ik(x-a)} & (x < a) \\ \mathrm{e}^{-ik(x-b)} + R_r\, \mathrm{e}^{ik(x-b)} & (x > b), \end{cases} \tag{2.3a}$$

$$\phi_2(x) = \begin{cases} \mathrm{e}^{ik(x-a)} + R_l\, \mathrm{e}^{-ik(x-a)} & (x < a) \\ \tau\, \mathrm{e}^{ik(x-b)} & (x > b). \end{cases} \tag{2.3b}$$

This defines the transmission coefficient τ, the right reflection coefficient R_r, and the left reflection coefficient R_l for the interval (a, b). We write them as $\tau(b, a; k)$, $R_r(b, a; k)$ and

---

[2] The Green function for the Fokker–Planck equation (1.2) is $G_F(x, y; k) = \mathrm{e}^{-[V(x)-V(y)]/2} G_S(x, y; k)$. When V_S does not make sense (see footnote 1), we need to define G_F first as the Green function for the Fokker–Planck equation, and then define $G_S = \mathrm{e}^{[V(x)-V(y)]/2} G_F$.
[3] This $G_S(k)$ becomes infinite where k^2 is an eigenvalue of the Schrödinger operator, and where k corresponds to a half-bound state. Elsewhere, this definition of $G_S(k)$ makes sense for real k, even on the continuous spectrum of the Schrödinger operator.

$R_l(b, a; k)$. The reflection coefficients for semi-infinite intervals are defined as the limit $a \to -\infty$ of $R_r(b, a; k)$ and the limit $b \to +\infty$ of $R_l(b, a; k)$. When k is a real number, it may happen that these limits do not exist. In such cases, we define

$$R_r(b, -\infty; k) \equiv \lim_{\epsilon \downarrow 0} \lim_{a \to -\infty} R_r(b, a; k + i\epsilon), \tag{2.4a}$$

$$R_l(\infty, a; k) \equiv \lim_{\epsilon \downarrow 0} \lim_{b \to +\infty} R_l(b, a; k + i\epsilon). \tag{2.4b}$$

In appendix G, it is shown that these limits indeed exist.

Let us define

$$S_r(x, k) \equiv \frac{R_r(x, -\infty; k)}{1 + R_r(x, -\infty; k)}, \qquad S_l(x, k) \equiv \frac{R_l(\infty; x; k)}{1 + R_l(\infty, x; k)}, \tag{2.5}$$

$$S(x, k) \equiv S_r(x, k) + S_l(x, k). \tag{2.6}$$

Then the Green function can be expressed in terms of this function S as [2]

$$G_S(x, y; k) = \frac{1}{2ik\sqrt{[1 - S(x, k)][1 - S(y, k)]}} \exp\left[ik(x - y) - ik \int_y^x S(z, k) \, \mathrm{d}z\right]. \tag{2.7}$$

The function $S(x, k)$ can also be expressed in terms of reflection coefficients for the Schrödinger equation. Let us consider the Schrödinger equation with the truncated potential which is set to be zero outside the interval (a, b)

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \bar{V}_S(x)\psi(x) = k^2\psi(x), \qquad \bar{V}_S(x) \equiv \begin{cases} 0 & (x < a) \\ V_S(x) & (a \leqslant x \leqslant b) \\ 0 & (b < x). \end{cases} \tag{2.8}$$

This equation has two solutions of the forms

$$\psi_1(x) = \begin{cases} \tau^S e^{-ik(x-a)} \\ e^{-ik(x-b)} + R_r^S e^{ik(x-b)} \end{cases} \qquad \psi_2(x) = \begin{cases} e^{ik(x-a)} + R_l^S e^{-ik(x-a)} & (x < a) \\ \tau^S e^{ik(x-b)} & (x > b). \end{cases} \tag{2.9}$$

The transmission coefficient $\tau^S(b, a; k)$ and the reflection coefficients $R_r^S(b, a; k)$, $R_l^S(b, a; k)$ are thus defined for the Schrödinger equation. We define R_r^S and R_l^S for semi-infinite intervals in the same way as (2.4). It can be shown that S_r and S_l defined by (2.5) are expressed in terms of R_r^S and R_l^S as

$$S_r(x, k) = \frac{R_r^S(x, -\infty)}{1 + R_r^S(x, -\infty)} - \frac{f(x)}{2ik}, \qquad S_l(x, k) = \frac{R_l^S(\infty, x)}{1 + R_l^S(\infty, x)} + \frac{f(x)}{2ik}. \tag{2.10}$$

(See appendix A for a proof.) The $f(x)$ in (2.10) cancels out when we substitute (2.10) into (2.6), and so S takes the same form as the expression in terms of R_r and R_l

$$S(x, k) = \frac{R_r^S(x, -\infty)}{1 + R_r^S(x, -\infty)} + \frac{R_l^S(\infty, x)}{1 + R_l^S(\infty, x)}. \tag{2.11}$$

(Incidentally, note that the right-hand side of (2.11) can also be written in terms of the Weyl–Titchmarsh m-function.)

4

3. Formulae for generalized reflection coefficients

Let us first make some definition. We define, for $n = 1, 2, 3, \ldots$ and $-\infty \leqslant a \leqslant b \leqslant +\infty$,

$$[\sigma_1, \sigma_2, \ldots, \sigma_n]_a^b \equiv \int \cdots \int_{a \leqslant z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n \leqslant b} dz_1 \cdots dz_n \exp\left[\sum_{j=1}^n \sigma_j V(z_j)\right], \tag{3.1}$$

where each σ_j is either $+1$ or -1. The integrals mean $\int_a^b dz_1 \int_{z_1}^b dz_2 \int_{z_2}^b dz_3 \cdots \int_{z_{n-1}}^b dz_n$. When $V_1 \equiv \lim_{x \to -\infty} V(x)$ is finite, we use the notation[4]

$$\langle -1, \sigma_2, \ldots, \sigma_n]_{-\infty}^b \equiv [-1, \sigma_2, \ldots, \sigma_n]_{-\infty}^b - e^{-2V_1}[+1, \sigma_2, \ldots, \sigma_n]_{-\infty}^b$$

$$= 2e^{-V_1} \int \cdots \int_{-\infty \leqslant z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n \leqslant b} dz_1 \cdots dz_n \sinh[V_1 - V(z_1)]$$

$$\times \exp\left[\sum_{j=2}^n \sigma_j V(z_j)\right]. \tag{3.2a}$$

For $n = 1$, this means $\langle -1]_{-\infty}^b \equiv 2e^{-V_1} \int_{-\infty}^b \sinh[V_1 - V(z)]\,dz$. Similarly, when $V_2 \equiv \lim_{x \to +\infty} V(x)$ is finite,

$$[\sigma_1, \ldots, \sigma_{n-1}, -1\rangle_a^\infty \equiv [\sigma_1, \ldots, \sigma_{n-1}, -1]_a^\infty - e^{-2V_2}[\sigma_1, \ldots, \sigma_{n-1}, +1]_a^\infty$$

$$= 2e^{-V_2} \int \cdots \int_{a \leqslant z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n \leqslant \infty} dz_1 \cdots dz_n \sinh[V_2 - V(z_n)]$$

$$\times \exp\left[\sum_{j=1}^{n-1} \sigma_j V(z_j)\right]. \tag{3.2b}$$

The conditions for the existence of these integrals will be discussed later.

In the formalism of [3], we deal with the generalized scattering coefficients, which are defined with an additional variable W as

$$\bar{R}_r(x, y; W; k) \equiv \frac{R_r(x, y; k) - \xi(x, W)}{1 - \xi(x, W)R_r(x, y; k)}, \tag{3.3a}$$

$$\bar{R}_l(x, y; W; k) \equiv R_l(x, y; k) + \frac{\xi(x, W)\tau^2(x, y; k)}{1 - \xi(x, W)R_r(x, y; k)}, \tag{3.3b}$$

$$\bar{\tau}(x, y; W; k) \equiv \frac{\gamma(x, W)\tau(x, y; k)}{1 - \xi(x, W)R_r(x, y; k)}, \tag{3.3c}$$

where

$$\xi(x, W) \equiv \tanh\frac{W - V(x)}{2}, \qquad \gamma(x, W) \equiv \sqrt{1 - \xi^2} = \text{sech}\frac{W - V(x)}{2}. \tag{3.4}$$

The original scattering coefficients R_r, R_l and τ are recovered from $\bar{R}_r$, $\bar{R}_l$ and $\bar{\tau}$ by setting $W = V(x)$. We define the operator $\mathcal{L}$, which acts on functions of x and W, as

$$\mathcal{L}g(x, W) \equiv 2\int_{-\infty}^x \left(\cosh[W - V(z)] + \sinh[W - V(z)]\frac{\partial}{\partial W}\right)g(z, W)\,dz. \tag{3.5}$$

The low-energy expansion of $\bar{R}_r$ for semi-infinite intervals was studied in [3]. According to the formulae derived there, we have

$$\bar{R}_r(x, -\infty; W; k) = \bar{r}_0 + ik\bar{r}_1 + (ik)^2\bar{r}_2 + \cdots + (ik)^N\bar{r}_N + \bar{\rho}_N, \tag{3.6}$$

---

[4] The relation with the notation used in [3] is $e^{V_1}\langle -1, \ldots\,]_{-\infty}^b = (\pm, \ldots\,]_{-\infty}^b$.

where[5]

$$\bar{r}_0 = -\tanh \frac{W - V(-\infty)}{2}, \tag{3.7}$$

$$\bar{r}_n(x, W) = \mathcal{L}^n[\bar{r}_0 + \xi(x, W)] \qquad (n \geqslant 1), \tag{3.8}$$

$$\bar{\rho}_N = (\mathrm{i}k)^{N+1} \int_{-\infty}^{x} \frac{\bar{\tau}^2(x, z; W; k)}{1 - \bar{R}_l^2(x, z; W; k)} \bar{r}'_{N+1}(z, \bar{\lambda}) \, \mathrm{d}z, \tag{3.9}$$

$$\bar{r}'_n(x, W) \equiv \frac{\partial}{\partial x} \bar{r}_n(x, W), \qquad \bar{\lambda} \equiv V(z) + \log \frac{1 + \bar{R}_l(x, z; W; k)}{1 - \bar{R}_l(x, z; W; k)}. \tag{3.10}$$

The integer $N \geqslant 0$ is arbitrary as long as $\bar{r}_0, \ldots, \bar{r}_N$ are finite. Let us derive the expressions for $\bar{r}_n$ and $\bar{\rho}_N$ in terms of the integrals defined by (3.1) and (3.2). We consider the three cases, (a) $V(-\infty) = V_1$ (finite), (b) $V(-\infty) = +\infty$ and (c) $V(-\infty) = -\infty$. We let the superscripts a, b and c stand for the cases (a), (b) and (c), respectively.

(a) $V(-\infty) = V_1$.

In this case we have

$$\bar{R}_r = \bar{r}_0^a + \mathrm{i}k\bar{r}_1^a + (\mathrm{i}k)^2\bar{r}_2^a + \cdots + (\mathrm{i}k)^N\bar{r}_N^a + \bar{\rho}_N^a, \tag{3.11}$$

$$\bar{r}_0^a = -\tanh \frac{W - V_1}{2}. \tag{3.12}$$

We can calculate $\bar{r}_n^a$ by substituting (3.12) into (3.8). The details of the calculation are given in appendix B. As a result, we obtain, for $n \geqslant 1$,

$$\bar{r}_n^a(x, W) = \sum_{\{\sigma_1, \ldots, \sigma_{n-1}\}} D_{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}}(W)\langle -1, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}]_{-\infty}^{x}, \tag{3.13}$$

where $\sum_{\{\sigma_1, \ldots, \sigma_{n-1}\}} = \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_{n-1} = \pm 1}$, and

$$D_{\sigma_1, \ldots, \sigma_{n-1}}(W) \equiv 2 \sum_{m=1}^{\infty} (-1)^{m+1} m P_{\sigma_1, \ldots, \sigma_{n-1}}^{(m)} \mathrm{e}^{(1-m)V_1} \mathrm{e}^{(m-\Lambda)W}, \quad \Lambda \equiv \sum_{i=1}^{n-1} \sigma_i, \tag{3.14}$$

$$P_{\sigma_1, \ldots, \sigma_{n-1}}^{(m)} \equiv \prod_{j=1}^{n-1} \left[\left(m - \sum_{i=1}^{j} \sigma_i \right)(-\sigma_j) \right]. \tag{3.15}$$

Substituting (3.13) into (3.9), the expression for the remainder term is obtained as

$$\bar{\rho}_N^a = 2(\mathrm{i}k)^{N+1} \sum_{\{\sigma_1, \ldots, \sigma_N\}} \int_{-\infty}^{x} \mathrm{d}z \frac{\bar{\tau}^2}{1 - \bar{R}_l^2} \langle -1, \sigma_1, \sigma_2, \ldots, \sigma_{N-1}]_{-\infty}^{z} \mathrm{e}^{\sigma_N V(z)}$$

$$\times \sum_{m=1}^{\infty} (-1)^{m+1} m P_{\sigma_1, \ldots, \sigma_N}^{(m)} \mathrm{e}^{(1-m)V_1} \mathrm{e}^{(m-\Lambda)V(z)} \left(\frac{1 + \bar{R}_l}{1 - \bar{R}_l} \right)^{m-\Lambda}. \tag{3.16}$$

(Here $\Lambda = \sum_{i=1}^{N} \sigma_i$.) In (3.16), $\bar{\tau}$ and $\bar{R}_l$ stand for $\bar{\tau}(x, z; W; k)$ and $\bar{R}_l(x, z; W; k)$, respectively. (If $N = 0$, the expression $\langle -1, \ldots, \sigma_{N-1}]_{-\infty}^{z} \mathrm{e}^{\sigma_N V(z)}$ in (3.16) is replaced

---

[5] For real k, the integral in (3.9) should be understood as $\lim_{\epsilon \downarrow 0} \lim_{y \to -\infty} \int_{y}^{x}$ with k replaced by $k + \mathrm{i}\epsilon$ in the integrand, if necessary.

by $2e^{-V_1}\sinh[V_1 - V(z)]$. Then $\Lambda = 0$ and $P^{(m)} = 1$.) For each N, the right-hand side of (3.16) can be written in a compact form without the infinite sum over m (see appendix B). For example, defining $\zeta \equiv \tanh\{[V_1 - V(z)]/2\}$, we can write $\bar{\rho}_0^a$ and $\bar{\rho}_1^a$ as

$$\bar{\rho}_0^a = 2\mathrm{i}k \int_{-\infty}^{x} \frac{\zeta \bar{\tau}^2}{(1 - \zeta \bar{R}_l)^2}\, \mathrm{d}z,$$

$$\bar{\rho}_1^a = (2\mathrm{i}k)^2\, \mathrm{e}^{-V_1} \int_{-\infty}^{x} \frac{(1 - \zeta^2)(1 + \zeta \bar{R}_l)\bar{\tau}^2}{(1 - \zeta \bar{R}_l)^3} \langle -1]_{-\infty}^{z}\, \mathrm{d}z.$$

(3.17)

Equation (3.11) makes sense if and only if $\bar{r}_0^a, \bar{r}_1^a, \dots, \bar{r}_N^a$ all exist as finite quantities. (By construction, the remainder term $\bar{\rho}_N^a$ is automatically finite if all $\bar{r}_n^a$ are finite, since $\bar{R}_r$ itself is finite.) Since $\bar{r}_n^a (n \geqslant 1)$ has the form of (3.13), it is necessary that $V(x)$ tend to V_1 fast enough, for otherwise the integrals defined by (3.2) do not exist. We can show that $\bar{r}_n^a$ is finite if $V - V_1 \in F_{n-1}^{(-)}$ (see appendix C). Therefore, $\bar{r}_0^a, \bar{r}_1^a, \dots, \bar{r}_N^a$ are all finite, and hence the expression (3.11) makes sense, if $V - V_1 \in F_{N-1}^{(-)}$.

(b) $V(-\infty) = +\infty$.

Next, we consider the case $V(-\infty) = +\infty$. We write

$$\bar{R}_r = \bar{r}_0^b + \mathrm{i}k\bar{r}_1^b + (\mathrm{i}k)^2\bar{r}_2^b + \cdots + (\mathrm{i}k)^N\bar{r}_N^b + \bar{\rho}_N^b. \tag{3.18}$$

Unlike (3.12), the first term $\bar{r}_0^b$, which is obtained from (3.7), is independent of W

$$\bar{r}_0^b = 1. \tag{3.19}$$

The higher order coefficients are obtained by substituting (3.19) into (3.8). The calculation is easier in this case (see appendix B), and we obtain

$$\bar{r}_n^b(x, W) = \sum_{\{\sigma_1, \dots, \sigma_{n-1}\}} 2P_{\sigma_1, \sigma_2, \dots, \sigma_{n-1}}^{(1)}\, \mathrm{e}^{(1-\Lambda)W}[-1, \sigma_1, \sigma_2, \dots, \sigma_{n-1}]_{-\infty}^{x}, \tag{3.20}$$

$$\bar{\rho}_N^b = 2(\mathrm{i}k)^{N+1} \sum_{\{\sigma_1, \dots, \sigma_N\}} \int_{-\infty}^{x} \mathrm{d}z\, \frac{\bar{\tau}^2}{1 - \bar{R}_l^2}[-1, \sigma_1, \sigma_2, \dots, \sigma_{N-1}]_{-\infty}^{z}\, \mathrm{e}^{\sigma_N V(z)}$$

$$\times P_{\sigma_1, \dots, \sigma_N}^{(1)}\, \mathrm{e}^{(1-\Lambda)V(z)}\left(\frac{1 + \bar{R}_l}{1 - \bar{R}_l}\right)^{1-\Lambda}. \tag{3.21}$$

(As before, $\Lambda = \sum_{i=1}^{n-1}\sigma_i$ in (3.20) and $\Lambda = \sum_{i=1}^{N}\sigma_i$ in (3.21).)

For (3.20) to exist as a finite quantity, it is necessary that $\mathrm{e}^{-V(x)}$ tend to 0 fast enough as $x \to -\infty$. It can be shown that $\bar{r}_n^b$ is finite if $\mathrm{e}^{-V} \in F_{n-1}^{(-)}$ (see appendix C). So the expression (3.18) makes sense if $\mathrm{e}^{-V} \in F_{N-1}^{(-)}$.

(c) $V(-\infty) = -\infty$.

The expressions for the case $V(-\infty) = -\infty$ can be obtained in the same way. We have

$$\bar{R}_r = \bar{r}_0^c + \mathrm{i}k\bar{r}_1^c + (\mathrm{i}k)^2\bar{r}_2^c + \cdots + (\mathrm{i}k)^N\bar{r}_N^c + \bar{\rho}_N^c, \qquad \bar{r}_0^c = -1, \tag{3.22}$$

$$\bar{r}_n^c(x, W) = -\sum_{\{\sigma_1, \dots, \sigma_{n-1}\}} 2P_{\sigma_1, \sigma_2, \dots, \sigma_{n-1}}^{(1)}\, \mathrm{e}^{-(1-\Lambda)W}[+1, -\sigma_1, -\sigma_2, \dots, -\sigma_{n-1}]_{-\infty}^{x}. \tag{3.23}$$

4. The low-energy expansion of S_r

The function S_r defined by (2.5) can be extracted from $\bar{R}_r$ as

$$S_r(x, k) = \lim_{\xi \to -1} \frac{\xi + \bar{R}_r}{1 - \xi^2} = \lim_{W \to -\infty} \frac{\xi(x, W) + \bar{R}_r(x, -\infty; W; k)}{1 - [\xi(x, W)]^2}. \tag{4.1}$$

Since $\xi(x, W) \sim -1 + 2e^{W-V(x)}$ and $1 - [\xi(x, W)]^2 \sim 4e^{W-V(x)}$ as $e^W \to 0$, we may write

$$S_r(x, k) - \tfrac{1}{2} = \tfrac{1}{4} \lim_{W \to -\infty} e^{-W+V(x)}[-1 + \bar{R}_r(x, -\infty; W; k)]. \tag{4.2}$$

(We have moved the $\frac{1}{2}$ to the left-hand side for convenience.) As in the last section, we consider the three cases (a), (b) and (c).

(a) $V(-\infty) = V_1$.
Substituting (3.11) into (4.2), we obtain

$$S_r(x, k) - \tfrac{1}{2} = a_0^R + ik a_1^R + (ik)^2 a_2^R + \cdots + (ik)^N a_N^R + \delta_N^{aR}, \tag{4.3}$$

$$a_0^R = \tfrac{1}{4} \lim_{W \to -\infty} e^{-W+V(x)}(\bar{r}_0^a - 1), \qquad a_n^R = \tfrac{1}{4} \lim_{W \to -\infty} e^{-W+V(x)} \bar{r}_n^a \qquad (n \geqslant 1), \tag{4.4}$$

$$\delta_N^{aR} = \tfrac{1}{4} \lim_{W \to -\infty} e^{-W+V(x)} \bar{\rho}_N^a. \tag{4.5}$$

From (3.12) and the first equation of (4.4), we have

$$a_0^R(x) = -\tfrac{1}{2} e^{-V_1} e^{V(x)}. \tag{4.6a}$$

For $n \geqslant 1$, we obtain a_n^R by substituting (3.13) into the second equation of (4.4). We can see from (3.15) that $P_{\sigma_1,\ldots,\sigma_{n-1}}^{(m)} = 0$ if $m \leqslant \Lambda$. So $m - \Lambda \geqslant 1$ in (3.14). Taking the limit $W \to -\infty$ of $e^{-W} \bar{r}_n^a$ amounts to picking out the terms with $m = \Lambda + 1$. This gives

$$a_n^R(x) = \sum_{\{\sigma_1,\ldots,\sigma_{n-1}\}} \frac{(-1)^\Lambda (\Lambda + 1)}{2} P_{\sigma_1,\sigma_2,\ldots,\sigma_{n-1}}^{(\Lambda+1)} e^{-\Lambda V_1} \langle -1, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}]_{-\infty}^x e^{V(x)}. \tag{4.6b}$$

The explicit expressions for the first few n are

$$a_0^R(x) = -\tfrac{1}{2} e^{-V_1} e^{V(x)}, \qquad a_1^R(x) = \tfrac{1}{2} \langle -]_{-\infty}^x e^{V(x)}, \qquad a_2^R(x) = e^{-V_1} \langle -+]_{-\infty}^x e^{V(x)},$$

$$a_3^R(x) = \big\{ 3e^{-2V_1} \langle -++]_{-\infty}^x - \langle --+]_{-\infty}^x \big\} e^{V(x)},$$

$$a_4^R(x) = \big\{ 12e^{-3V_1} \langle -+++]_{-\infty}^x - 2e^{-V_1} \langle -+-+]_{-\infty}^x - 6e^{-V_1} \langle --++]_{-\infty}^x \big\} e^{V(x)}, \tag{4.7}$$

$$a_5^R(x) = \big\{ 60e^{-4V_1} \langle -++++]_{-\infty}^x - 6e^{-2V_1} \langle -++-+]_{-\infty}^x - 18e^{-2V_1} \langle -+-++]_{-\infty}^x$$

$$\qquad - 36e^{-2V_1} \langle --+++]_{-\infty}^x + 2\langle --+-+]_{-\infty}^x + 6\langle ---++]_{-\infty}^x \big\} e^{V(x)},$$

where we have used the shorthand notation $\langle -]_{-\infty}^x$, $\langle -+]_{-\infty}^x$, etc for $\langle -1]_{-\infty}^x$, $\langle -1, +1]_{-\infty}^x$, etc.

Obviously $|a_n^R| < \infty$ if $V - V_1 \in F_{n-1}^{(-)}$. (This is the same as the condition for $|\bar{r}_n^a| < \infty$ discussed in the previous section.) So, equation (4.3) makes sense if $V - V_1 \in F_{N-1}^{(-)}$.

(b) $V(-\infty) = +\infty$.
In this case, we substitute (3.18) into (4.2). This leads us to consider the limits

$$b_0^R = \tfrac{1}{4} \lim_{W \to -\infty} e^{-W+V(x)}(\bar{r}_0^b - 1), \qquad b_n^R = \tfrac{1}{4} \lim_{W \to -\infty} e^{-W+V(x)} \bar{r}_n^b \qquad (n \geqslant 1), \tag{4.8}$$

$$\delta_N^{bR} = \tfrac{1}{4} \lim_{W \to -\infty} e^{-W+V(x)} \bar{\rho}_N^b. \tag{4.9}$$

From (3.19) and (4.8), we have $b_0^R = 0$. For $n \geqslant 1$, we can see that only the terms with $\Lambda = 0$ in (3.20) survive in the limit of (4.8). Since Λ is an odd number for even n, it follows that $b_n^R = 0$ for any even n. So, in this case we have the expression

$$S_r(x, k) - \tfrac{1}{2} = \mathrm{i}k b_1^R + (\mathrm{i}k)^3 b_3^R + (\mathrm{i}k)^5 b_5^R + \cdots + (\mathrm{i}k)^{2M+1} b_{2M+1}^R + \delta_{2M+1}^{bR}, \tag{4.10}$$

which has only odd powers of k. Since $\delta_{n-1}^{bR} = (\mathrm{i}k)^n b_n^R + \delta_n^{bR}$, we have $\delta_n^{bR} = \delta_{n-1}^{bR}$ for even n. The coefficients b_n^R are easily obtained as

$$b_n^R(x) = \frac{1}{2} \sum_{\substack{\{\sigma_1, \ldots, \sigma_{n-1}\} \\ \Lambda = 0}} P_{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}}^{(1)} [-1, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}]_{-\infty}^x \, \mathrm{e}^{V(x)}, \tag{4.11}$$

where the sum is taken with the constraint $\Lambda = \sum_{i=1}^{n-1} \sigma_i = 0$. Using the shorthand notation $[-]_{-\infty}^x$, etc for $[-1]_{-\infty}^x$, etc, we can explicitly write, for the first few n,

$$b_1^R(x) = \tfrac{1}{2}[-]_{-\infty}^x \, \mathrm{e}^{V(x)}, \qquad b_3^R(x) = -[- - +]_{-\infty}^x \, \mathrm{e}^{V(x)},$$

$$b_5^R(x) = \left\{ 6[- - - + +]_{-\infty}^x + 2[- - + - +]_{-\infty}^x \right\} \mathrm{e}^{V(x)},$$

$$b_7^R(x) = -\left\{ 72[- - - - + + +]_{-\infty}^x + 36[- - - + - + +]_{-\infty}^x + 12[- - - + + - +]_{-\infty}^x \right. \tag{4.12}$$
$$\left. + 12[- - + - - + +]_{-\infty}^x + 4[- - + - + - +]_{-\infty}^x \right\} \mathrm{e}^{V(x)}.$$

Note that (4.12) can be formally obtained from (4.7) by letting $V_1 \to +\infty$ and $\langle - \cdots \rangle_{-\infty}^x \to [- \cdots]_{-\infty}^x$.

It is obvious that $\left| b_n^R \right| < \infty$ if $\mathrm{e}^{-V} \in F_{n-1}^{(-)}$, which is the same as the condition for $\left| \bar{r}_n^b \right| < \infty$ studied in the previous section. Therefore, (4.10) makes sense if $\mathrm{e}^{-V} \in F_{2M}^{(-)}$.

(c) $V(-\infty) = -\infty$.

In this case, we cannot obtain the expansion of $S_r - \tfrac{1}{2}$ by substituting (3.22) into (4.2), since $\lim_{W \to -\infty} \mathrm{e}^{-W}(\bar{r}_0^c - 1)$ is not finite. Instead, the expansion of $\left(S_r - \tfrac{1}{2} \right)^{-1}$ can be obtained in the same way as in case (b). As shown in appendix D, we have

$$\left[S_r(x, k) - \tfrac{1}{2} \right]^{-1} = 4 \left[\mathrm{i}k \tilde{b}_1^R + (\mathrm{i}k)^3 \tilde{b}_3^R + (\mathrm{i}k)^5 \tilde{b}_5^R + \cdots + (\mathrm{i}k)^{2M+1} \tilde{b}_{2M+1}^R + \delta_{2M+1}^{\tilde{b}R} \right], \tag{4.13}$$

where $\tilde{b}_n^R$ and $\delta_N^{\tilde{b},R}$ are the quantities obtained form b_n^R and δ_N^{bR} by changing the sign of the potential V. Namely,

$$\tilde{b}_n^R(x) = \frac{1}{2} \sum_{\substack{\{\sigma_1, \ldots, \sigma_{n-1}\} \\ \Lambda = 0}} P_{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}}^{(1)} [+1, -\sigma_1, -\sigma_2, \ldots, -\sigma_{n-1}]_{-\infty}^x \, \mathrm{e}^{-V(x)}, \tag{4.14}$$

or, explicitly, $\tilde{b}_1^R(x) = \tfrac{1}{2}[+]_{-\infty}^x \, \mathrm{e}^{-V(x)}$, $\tilde{b}_3^R(x) = -[+ + -]_{-\infty}^x \, \mathrm{e}^{-V(x)}$, etc. From (4.13), the expansion of $S_r - \tfrac{1}{2}$ is obtained as

$$S_r(x, k) - \frac{1}{2} = (\mathrm{i}k)^{-1} \gamma_{-1}^R + \mathrm{i}k \gamma_1^R + (\mathrm{i}k)^3 \gamma_3^R + \cdots + (\mathrm{i}k)^{2M+1} \gamma_{2M+1}^R + \delta_{2M+1}^{\gamma R}, \tag{4.15}$$

$$\gamma_{-1}^R = \frac{1}{4\tilde{b}_1^R}, \qquad \gamma_1^R = -\frac{\tilde{b}_3^R}{4(\tilde{b}_1^R)^2}, \qquad \gamma_3^R = \frac{1}{4(\tilde{b}_1^R)^3} \left[(\tilde{b}_3^R)^2 - \tilde{b}_5^R \tilde{b}_1^R \right], \quad \text{etc.} \tag{4.16}$$

Obviously $\gamma_n^R = 0$ and $\delta_n^{\gamma R} = \delta_{n-1}^{\gamma R}$ for even n.

We know that $\left| \tilde{b}_n^R \right| < \infty$ if $\mathrm{e}^V \in F_{n-1}^{(-)}$. (This is obtained from the condition for $\left| b_n^R \right| < \infty$ by changing the sign of V.) Since γ_n^R is expressed in terms of $\tilde{b}_m^R$ with $m \leqslant n + 2$, we can see that $\left| \gamma_n^R \right| < \infty$ if $\mathrm{e}^V \in F_{n+1}^{(-)}$, and hence that (4.15) makes sense if $\mathrm{e}^V \in F_{2M+2}^{(-)}$.

The expressions for S_l can be obtained in the parallel way. We consider the three cases, where $V(+\infty)$ is finite ($= V_2$), $+\infty$ or $-\infty$. The results are as follows:

(a) If $V(+\infty) = V_2$,

$$S_l(x, k) - \frac{1}{2} = a_0^L + ik a_1^L + (ik)^2 a_2^L + \cdots + (ik)^N a_N^L + \delta_N^{aL}, \tag{4.17}$$

$$a_0^L(x) = -\frac{1}{2} e^{-V_2} e^{V(x)}, \tag{4.18a}$$

$$a_n^L(x) = \sum_{\{\sigma_1,...,\sigma_{n-1}\}} \frac{(-1)^\Lambda (\Lambda + 1)}{2} P_{\sigma_1,\sigma_2,...,\sigma_{n-1}}^{(\Lambda+1)} e^{-\Lambda V_2} [\sigma_{n-1}, \ldots, \sigma_2, \sigma_1, -1\rangle_x^\infty e^{V(x)}. \tag{4.18b}$$

(b) If $V(+\infty) = +\infty$,

$$S_l(x, k) - \frac{1}{2} = ik b_1^L + (ik)^3 b_3^L + (ik)^5 b_5^L + \cdots + (ik)^{2M+1} b_{2M+1}^L + \delta_{2M+1}^{bL} \tag{4.19}$$

$$b_n^L(x) = \frac{1}{2} \sum_{\substack{\{\sigma_1,...,\sigma_{n-1}\} \\ \Lambda=0}} P_{\sigma_1,\sigma_2,...,\sigma_{n-1}}^{(1)} [\sigma_{n-1}, \ldots, \sigma_2, \sigma_1, -1]_x^\infty e^{V(x)}. \tag{4.20}$$

(c) If $V(+\infty) = -\infty$,

$$\left[S_l(x, k) - \frac{1}{2} \right]^{-1} = 4 \left[ik \tilde{b}_1^L + (ik)^3 \tilde{b}_3^L + (ik)^5 \tilde{b}_5^L + \cdots + (ik)^{2M+1} \tilde{b}_{2M+1}^L + \delta_{2M+1}^{\tilde{b}L} \right], \tag{4.21}$$

$$S_l(x, k) - \frac{1}{2} = (ik)^{-1} \gamma_{-1}^L + ik \gamma_1^L + (ik)^3 \gamma_3^L + \cdots + (ik)^{2M+1} \gamma_{2M+1}^L + \delta_{2M+1}^{\gamma L}, \tag{4.22}$$

$$\tilde{b}_n^L(x) = \frac{1}{2} \sum_{\substack{\{\sigma_1,...,\sigma_{n-1}\} \\ \Lambda=0}} P_{\sigma_1,\sigma_2,...,\sigma_{n-1}}^{(1)} [-\sigma_{n-1}, \ldots, -\sigma_2, -\sigma_1, +1]_x^\infty e^{-V(x)}. \tag{4.23}$$

5. Low-energy expansion of the Green function

The expansion of the function S (equation (2.6)) is obtained by adding the expressions for S_r (equation (4.3), (4.10) or (4.15)) and S_l ((4.17), (4.19) or (4.22)). We can derive the expansion of G_S by substituting this into (2.7). We study the six cases listed in the introduction.

Case (i): $V(-\infty) = V_1$, $V(+\infty) = V_2$.

Adding (4.3) and (4.17) together, we obtain the expansion of $S - 1$ as

$$S - 1 = s_0 + ik s_1 + (ik)^2 s_2 + \cdots + (ik)^N s_N + \delta_N, \tag{5.1}$$

$$s_n = a_n^R + a_n^L, \qquad \delta_N = \delta_N^{aR} + \delta_N^{aL}. \tag{5.2}$$

Substituting (5.1) into (2.7) we can derive the expansion

$$G_S(x, y; k) = (ik)^{-1} g_{-1} + g_0 + ik g_1 + (ik)^2 g_2 + \cdots + (ik)^N g_N + \Delta_N, \tag{5.3}$$

$$g_{-1} = \frac{1}{2\sqrt{s_0(x)s_0(y)}}, \qquad g_0 = \left(q_1(x,y) - \frac{s_1(x)}{2s_0(x)} - \frac{s_1(y)}{2s_0(y)}\right)g_{-1}, \quad \text{etc}, \tag{5.4}$$

where we have defined

$$q_n(x,y) \equiv -\int_y^x s_{n-1}(z)\,\mathrm{d}z. \tag{5.5}$$

(The remainder term Δ_N will be discussed in the following section.) From (4.6), (4.18), (5.2) and (5.4), we obtain the explicit expressions for g_{-1} and g_1,

$$g_{-1} = \frac{\mathrm{e}^{-[V(x)+V(y)]/2}}{\mathrm{e}^{-V_1} + \mathrm{e}^{-V_2}}, \tag{5.6a}$$

$$g_0 = \frac{\mathrm{e}^{-[V(x)+V(y)]/2}}{2}\left\{\frac{1}{(\mathrm{e}^{-V_1}+\mathrm{e}^{-V_2})^2}\left(\langle-]_{-\infty}^x + [-\rangle_x^\infty + \langle-]_{-\infty}^y + [-\rangle_y^\infty\right) + [+]_y^x\right\}, \tag{5.6b}$$

which are the same as the expressions derived in [17] by a different method.

Case (ii): $V(-\infty) = V_1$, $V(+\infty) = +\infty$.

In this case, the expansion of $S - 1$ has the same form as (5.1), with

$$s_n = a_n^{\mathrm{R}} + b_n^{\mathrm{L}}, \qquad \delta_n = \delta_n^{a\mathrm{R}} + \delta_n^{b\mathrm{L}}. \tag{5.7}$$

Since $b_n^{\mathrm{R}} = 0$ and $\delta_n^{b\mathrm{R}} = \delta_{n-1}^{b\mathrm{R}}$ for even n, equations (5.7) read $s_n = a_n^{\mathrm{R}}$ and $\delta_n = \delta_n^{a\mathrm{R}} + \delta_{n-1}^{b\mathrm{L}}$ for even n. The expansion of G_{S}, too, has the same form as (5.3) with (5.4). Substituting (4.6) and (4.20), we obtain

$$g_{-1} = \mathrm{e}^{-[V(x)+V(y)]/2}\,\mathrm{e}^{V_1}, \tag{5.8a}$$

$$g_0 = \frac{\mathrm{e}^{-[V(x)+V(y)]/2}}{2}\left\{\mathrm{e}^{2V_1}\left(\langle-]_{-\infty}^x + [-]_x^\infty + \langle-]_{-\infty}^y + [-]_y^\infty\right) + [+]_y^x\right\}. \tag{5.8b}$$

Formally, (5.8) can also be obtained from (5.6) by letting $V_2 \to \infty$ and $[-\rangle_a^\infty \to [-]_a^\infty$.

Case (iii): $V(-\infty) = V_1$, $V(+\infty) = -\infty$.

When $V(+\infty) = -\infty$, the expansion of $S - 1$, which is obtained from (4.3) and (4.22), contains a term of order $1/k$

$$S - 1 = (\mathrm{i}k)^{-1}s_{-1} + s_0 + \mathrm{i}ks_1 + \cdots + (\mathrm{i}k)^N s_N + \delta_N, \tag{5.9}$$

$$s_{-1} = \gamma_{-1}^{\mathrm{L}}, \qquad s_n = a_n^{\mathrm{R}} + \gamma_n^{\mathrm{L}} \quad (n \geqslant 0), \qquad \delta_n = \delta_n^{a\mathrm{R}} + \delta_n^{\gamma\mathrm{L}}. \tag{5.10}$$

When n is even, $s_n = a_n^{\mathrm{R}}$ since $\gamma_n^{\mathrm{L}} = 0$. Corresponding to (5.9), the expansion of G_{S} lacks the term of order $1/k$ which was present in (5.3)

$$G_{\mathrm{S}}(x,y;k) = g_0 + \mathrm{i}kg_1 + (\mathrm{i}k)^2 g_2 + \cdots + (\mathrm{i}k)^N g_N + \Delta_N, \tag{5.11}$$

$$g_0 = \frac{-\exp[q_0(x,y)]}{2\sqrt{s_{-1}(x)s_{-1}(y)}}, \qquad g_1 = \left(q_1(x,y) - \frac{s_0(x)}{2s_{-1}(x)} - \frac{s_0(y)}{2s_{-1}(y)}\right)g_0, \quad \text{etc}, \tag{5.12}$$

where q_0 and q_1 are defined by (5.5). For $n = -1$ and 0, we have $s_{-1}(x) = \mathrm{e}^{V(x)}/(2[+]_x^\infty)$ and $s_0(x) = -\mathrm{e}^{V(x)-V_1}/2$. Note that $\frac{\mathrm{d}}{\mathrm{d}z}[+]_z^\infty = -\mathrm{e}^{V(z)}$. Therefore,

$$\exp[q_0(x,y)] = \exp\left[\frac{1}{2}\int_y^x\left(\frac{\mathrm{d}}{\mathrm{d}z}\log[+]_z^\infty\right)\mathrm{d}z\right] = \sqrt{\frac{[+]_x^\infty}{[+]_y^\infty}}. \tag{5.13}$$

Also using $q_1(x, y) = -\int_y^x s_0(z)\,dz = \frac{1}{2}\,e^{-V_1}[+]_y^x$, from (5.12) we obtain

$$g_0 = -e^{-[V(x)+V(y)]/2}[+]_x^\infty, \qquad g_1 = -e^{-[V(x)+V(y)]/2}\,e^{-V_1}[+]_x^\infty[+]_y^\infty. \qquad (5.14)$$

Case (iv): $V(-\infty) = +\infty$, $V(+\infty) = +\infty$.

When both $V(\pm\infty)$ are $+\infty$, the expansion of $S - 1$ has only odd powers of k

$$S - 1 = iks_1 + (ik)^3 s_3 + (ik)^5 s_5 + \cdots + (ik)^{2M+1} s_{2M+1} + \delta_{2M+1}, \qquad (5.15)$$

$$s_n = b_n^R + b_n^L, \qquad \delta_n = \delta_n^{bR} + \delta_n^{bL}. \qquad (5.16)$$

(Note that $s_n = 0$ and $\delta_n = \delta_{n-1}$ for any even n.) The corresponding expression for G_S begins with the term of order $1/k^2$, and has only even powers of k

$$G_S(x, y; k) = (ik)^{-2} g_{-2} + g_0 + (ik)^2 g_2 + (ik)^4 g_4 + \cdots + (ik)^{2M} g_{2M} + \Delta_{2M}, \qquad (5.17)$$

$$g_{-2} = \frac{-1}{2\sqrt{s_1(x)s_1(y)}}, \qquad g_0 = \left(q_2(x, y) - \frac{s_3(x)}{2s_1(x)} - \frac{s_3(y)}{2s_1(y)}\right) g_{-2}, \quad \text{etc.} \qquad (5.18)$$

(In deriving (5.18) we choose the branch of the square root in (2.7) so that $\sqrt{-k^2} = -ik$.) For $n = 1$ and $n = 3$, the first equation of (5.16) reads

$$s_1(x) = \frac{1}{2}\,e^{V(x)}[-]_{-\infty}^\infty, \qquad s_3(x) = -e^{V(x)}\left([- - +]_{-\infty}^x + [+ - -]_x^\infty\right). \qquad (5.19)$$

(Note that $[-]_{-\infty}^x + [-]_x^\infty = [-]_{-\infty}^\infty$.) Hence $q_2(x, y) = -\frac{1}{2}[+]_y^x[-]_{-\infty}^\infty$. Substituting these expressions into (5.18) yields

$$g_{-2} = -e^{-[V(x)+V(y)]/2}\frac{1}{[-]_{-\infty}^\infty}, \qquad (5.20a)$$

$$g_0 = e^{-[V(x)+V(y)]/2}\left(-\frac{[- - +]_{-\infty}^x + [+ - -]_x^\infty + [- - +]_{-\infty}^y + [+ - -]_y^\infty}{\left([-]_{-\infty}^\infty\right)^2} + \frac{[+]_y^x}{2}\right). \qquad (5.20b)$$

Case (v): $V(-\infty) = +\infty$, $V(+\infty) = -\infty$.

In this case, too, the expansion of $S - 1$ has only odd powers of k, but now the series begins with the term of order $1/k$

$$S - 1 = (ik)^{-1} s_{-1} + iks_1 + (ik)^3 s_3 + \cdots + (ik)^{2M+1} s_{2M+1} + \delta_{2M+1}, \qquad (5.21)$$

$$s_{-1} = \gamma_{-1}^L, \qquad s_n = b_n^R + \gamma_n^L \quad (n \geqslant 0), \qquad \delta_n = \delta_n^{bR} + \delta_n^{\gamma L}. \qquad (5.22)$$

(For even n, we have $s_n = 0$ and $\delta_n = \delta_{n-1}$.) Correspondingly, the expansion of G_S begins with the term of order k^0

$$G_S(x, y; k) = g_0 + (ik)^2 g_2 + (ik)^4 g_4 + \cdots + (ik)^{2M} g_{2M} + \Delta_{2M}, \qquad (5.23)$$

$$g_0 = \frac{-\exp[q_0(x, y)]}{2\sqrt{s_{-1}(x)s_{-1}(y)}}, \qquad g_2 = \left(q_2(x, y) - \frac{s_1(x)}{2s_{-1}(x)} - \frac{s_1(y)}{2s_{-1}(y)}\right) g_0, \quad \text{etc.} \qquad (5.24)$$

We have $s_{-1}(x) = e^{V(x)}/\left(2[+]_x^\infty\right)$ and $s_1(x) = e^{V(x)}\left\{[- +]_x^\infty/\left([+]_x^\infty\right)^2 + \frac{1}{2}[-]_{-\infty}^x\right\}$. Since s_{-1} is the same as in case (iii), obviously g_0 is the same as (5.14), i.e.,

$$g_0 = -e^{-[V(x)+V(y)]/2}[+]_x^\infty. \qquad (5.25a)$$

12

To calculate q_2, we use $\int_y^x e^{V(z)}\left\{[-++]_z^\infty\big/\left([+]_z^\infty\right)^2\right\}dz = \int_y^x[-++]_z^\infty\frac{d}{dz}\left(1/[+]_z^\infty\right)dz$ and integrate by parts. Substituting this and s_{-1}, s_1 into the second equation of (5.24) gives

$$g_2 = e^{-[V(x)+V(y)]/2}\left\{\left([-]_{-\infty}^x[+]_x^\infty + [-]_{-\infty}^y[+]_y^x + [-+]_y^x\right)[+]_x^\infty + 2[-++]_x^\infty\right\}. \tag{5.25b}$$

Case (vi): $V(-\infty) = -\infty$, $V(+\infty) = -\infty$.

In this case, the expansion of $S-1$ has the same form as (5.21), where

$$s_n = \gamma_n^R + \gamma_n^L, \qquad \delta_n = \delta_n^{\gamma R} + \delta_n^{\gamma L}. \tag{5.26}$$

(As before, $s_n = 0$ and $\delta_n = \delta_{n-1}$ for even n.) The expansion of G_S is given by the same expression as (5.23) with (5.24). Substituting $s_{-1}(x) = e^{V(x)}[+]_{-\infty}^\infty\big/\left(2[+]_{-\infty}^x[+]_x^\infty\right)$ and $\exp[q_0(x,y)] = \sqrt{[+]_{-\infty}^y[+]_x^\infty\big/\left([+]_{-\infty}^x[+]_y^\infty\right)}$, we obtain

$$g_0 = -e^{-[V(x)+V(y)]/2}\frac{[+]_{-\infty}^y[+]_x^\infty}{[+]_{-\infty}^\infty}. \tag{5.27}$$

(The expression for g_2 does not become simpler than (5.24) in this case.)

Now we have derived the expansion of G_S for each of the six cases. To summarize, the expansion has the form of (5.3) (in cases (i) and (ii)), (5.11) (in case (iii)), (5.17) (in case (iv)) or (5.23) (in cases (v) and (vi)). These expressions make sense if and only if $|g_n| < \infty$ for any $n \leqslant N$ ($N \equiv 2M$ for (5.17) and (5.23)). The remainder term Δ_N is finite if all g_n are finite. As mentioned in the previous section, we know that

$$
\begin{aligned}
&\left|a_n^R\right| < \infty \quad \text{if} \quad V - V_1 \in F_{n-1}^{(-)}, &&\left|a_n^L\right| < \infty \quad \text{if} \quad V - V_2 \in F_{n-1}^{(+)},\\
&\left|b_n^R\right| < \infty \quad \text{if} \quad e^{-V} \in F_{n-1}^{(-)}, &&\left|b_n^L\right| < \infty \quad \text{if} \quad e^{-V} \in F_{n-1}^{(+)}, \tag{5.28}\\
&\left|\gamma_n^R\right| < \infty \quad \text{if} \quad e^V \in F_{n+1}^{(-)}, &&\left|\gamma_n^L\right| < \infty \quad \text{if} \quad e^V \in F_{n+1}^{(+)}.
\end{aligned}
$$

From (5.28), we can derive the sufficient conditions for $|g_N| < \infty$. These conditions are also sufficient for $|g_n| < \infty$ ($n < N$). As a result, we find that

in case (i), equation (5.3) makes sense if $V - V_1 \in F_N^{(-)}$ and $V - V_2 \in F_N^{(+)}$,

in case (ii), equation (5.3) makes sense if $V - V_1 \in F_N^{(-)}$ and $e^{-V} \in F_N^{(+)}$,

in case (iii), equation (5.11) makes sense if $V - V_1 \in F_{N-2}^{(-)}$ and $e^V \in F_N^{(+)}$,

in case (iv) equation (5.17) makes sense if $e^{-V} \in F_{2M+2}^{(-)}$ and $e^{-V} \in F_{2M+2}^{(+)}$, $\qquad$ (5.29)

in case (v), equation (5.23) makes sense if $e^{-V} \in F_{2M-2}^{(-)}$ and $e^V \in F_{2M}^{(+)}$,

in case (vi), equation (5.23) makes sense if $e^V \in F_{2M}^{(-)}$ and $e^V \in F_{2M}^{(+)}$.

(The conditions involving $F_n^{(\pm)}$ with $n < 0$ are interpreted as automatically satisfied.)

6. Behavior of Δ_N as $k \to 0$

The expansion of G_S to order k^N is meaningful as a low-energy expansion only if the remainder term satisfies $\Delta_N = o(k^N)$ as $k \to 0$. In this section, we study the conditions for this to hold. Substituting the expansion of $S-1$ into (2.7), we can easily see that

in cases (i) and (ii), $\qquad \Delta_{n-1} = o(k^{n-1})$ if $\delta_n = o(k^n)$ $(n \geqslant 0)$, $\qquad$ (6.1a)

in cases (iii), (v) and (vi), $\qquad \Delta_{n+1} = o(k^{n+1})$ if $\delta_n = o(k^n)$ $(n \geqslant -1)$, $\qquad$ (6.1b)

in case (iv), $\qquad \Delta_{n-3} = o(k^{n-3})$ if $\delta_n = o(k^n)$ $(n \geqslant 1)$. $\qquad$ (6.1c)

(We have used the fact that the integral in (2.7) and the limit $k \to 0$ are interchangeable, as can be easily shown.) By definition, δ_n is equal to (i) $\delta_n^{aR} + \delta_n^{aL}$, (ii) $\delta_n^{aR} + \delta_n^{bL}$, (iii) $\delta_n^{aR} + \delta_n^{\gamma L}$, (iv) $\delta_n^{bR} + \delta_n^{bL}$, (v) $\delta_n^{bR} + \delta_n^{\gamma L}$ or (vi) $\delta_n^{\gamma R} + \delta_n^{\gamma L}$, according to the six cases. So, the behavior of Δ_N as $k \to 0$ can be known from the behavior of δ_n^{aR}, δ_n^{bR}, etc.

The small-k behavior of δ_n^{aR}, δ_n^{bR}, etc can be studied using (3.16) and (3.21). The detailed analysis is given in appendix E. The result is

$$\delta_n^{aR} = o(k^n) \quad \text{if} \quad V - V_1 \in F_{n-1}^{(-)}, \qquad \delta_n^{aL} = o(k^n) \qquad \text{if} \quad V - V_2 \in F_{n-1}^{(+)}, \qquad (6.2a)$$

$$\delta_n^{bR} = o(k^n) \quad \text{if} \quad e^{-V} \in F_{n-1}^{(-)}, \qquad \delta_n^{bL} = o(k^n) \qquad \text{if} \quad e^{-V} \in F_{n-1}^{(+)}, \qquad (6.2b)$$

$$\delta_{n-1}^{\gamma R} = o(k^{n-1}) \quad \text{if} \quad e^V \in F_n^{(-)}, \qquad \delta_{n-1}^{\gamma L} = o(k^{n-1}) \quad \text{if} \quad e^V \in F_n^{(+)}. \qquad (6.2c)$$

(Here $n \geqslant 0$. For $n = 0$, the conditions involving $F_{n-1}^{(\pm)}$ should be interpreted as automatically satisfied.) From (6.1) and (6.2), we can conclude that

in case (i), $\quad \Delta_N = o(k^N) \quad \text{if} \quad V - V_1 \in F_N^{(-)} \quad \text{and} \quad V - V_2 \in F_N^{(+)} \quad (N \geqslant -1)$,

in case (ii), $\quad \Delta_N = o(k^N) \quad \text{if} \quad V - V_1 \in F_N^{(-)} \quad \text{and} \quad e^{-V} \in F_N^{(+)} \quad (N \geqslant -1)$,

in case (iii), $\quad \Delta_N = o(k^N) \quad \text{if} \quad V - V_1 \in F_{N-2}^{(-)} \quad \text{and} \quad e^V \in F_N^{(+)} \quad (N \geqslant 0)$,

in case (iv), $\quad \Delta_N = o(k^N) \quad \text{if} \quad e^{-V} \in F_{N+2}^{(-)} \quad \text{and} \quad e^{-V} \in F_{N+2}^{(+)} \quad (N \geqslant -2)$,

in case (v), $\quad \Delta_N = o(k^N) \quad \text{if} \quad e^{-V} \in F_{N-2}^{(-)} \quad \text{and} \quad e^V \in F_N^{(+)} \quad (N \geqslant 0)$,

in case (vi), $\quad \Delta_N = o(k^N) \quad \text{if} \quad e^V \in F_N^{(-)} \quad \text{and} \quad e^V \in F_N^{(+)} \quad (N \geqslant 0)$.

(6.3)

The conditions in (6.3) are exactly the same as the conditions in (5.29) (where $N = 2M$ for cases (vi), (v), (vi)). Therefore, the expansion to order k^N makes sense and is valid as an asymptotic expansion if these conditions are satisfied.

The marginal cases for the conditions of (6.3) are $V(z) \sim A + \beta/|z|^\alpha$ (where α, β are constants, and $A = V_1$ or V_2) and $V(z) \sim \alpha \log|z|$ as $z \to -\infty$ or $+\infty$. These cases correspond to $V_S(z) \sim C/|z|^{2+\alpha}$ and $V_S(z) \sim l(l+1)/|z|^2$ with $l \equiv \alpha/2$, respectively. Let α be a non-integer such that $0 \leqslant n < \alpha < n + 1$. Then, as shown in appendix E,

$$\delta_n^{aR} \sim Ck^\alpha \quad (k \to 0) \quad \text{if} \quad V(z) \sim V_1 + \frac{\beta}{|z|^\alpha} \quad (z \to -\infty), \qquad (6.4a)$$

$$\delta_n^{bR} \sim Ck^\alpha \quad (k \to 0) \quad \text{if} \quad V(z) \sim \alpha \log|z| \quad (z \to -\infty), \qquad (6.4b)$$

and similarly for δ_n^{aL} and δ_n^{bL} (where C is a certain constant). The behavior of Δ_N for the marginal cases can be easily known from (6.4) (and the corresponding expressions for δ_n^{aL} and δ_n^{bL}). For example, if $V - V_1 \in F_{N+1}^{(-)}$ and $V(z) \sim \alpha \log z$ as $z \to +\infty$ with $N+1 < \alpha < N+2$, then $\Delta_N \sim Ck^{\alpha-1}$ as $k \to 0$ (see example 5 of section 8).

7. Schrödinger equation with a potential vanishing at $x \to \pm\infty$

Suppose that $V_S(x)$ is given, and that $f(x)$ and $V(x)$ are yet unknown. Let $\psi_0(x)$ be a solution of (1.1) with $k = 0$. Then a function f satisfying (1.4) is obtained from ψ_0 as

$$f(x) = \frac{d}{dx} \log \psi_0(x). \qquad (7.1)$$

The Schrödinger equation (1.1) is equivalent to the Fokker–Planck equation (1.2) with (7.1), where ϕ is related to ψ by $\phi(x) = \psi_0(x)\psi(x)$. The Fokker–Planck potential is expressed

in terms of ψ_0 as $V(x) = -2\log\psi_0(x)$. (In order to make $V(x)$ finite for any finite x, the function ψ_0 needs to satisfy $\psi_0(x) > 0$ for any finite x.) A given Schrödinger equation can be thus transformed into a Fokker–Planck equation.

In our formalism, $V_S(x)$ need not be zero (or even finite) as $x \to \pm\infty$. But here we study a particular feature of the case where $V_S(x)$ tends to zero at both $x \to +\infty$ and $x \to -\infty$. For simplicity, we assume that there are no bound states.

Now we assume $V_S(\pm\infty) = 0$. Let $\psi_0^+(x)$ and $\psi_0^-(x)$ denote the solutions of (1.1) with $k = 0$, such that $\psi_0^+(x) \to 1$ as $x \to +\infty$ and $\psi_0^-(x) \to 1$ as $x \to -\infty$. We define

$$V_\pm(x) \equiv -2\log\psi_0^\pm(x), \qquad f_\pm(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\log\psi_0^\pm(x). \tag{7.2}$$

Obviously, $V_+(x) \to 0$ as $x \to +\infty$ and $V_-(x) \to 0$ as $x \to -\infty$. If ψ_0^+ and ψ_0^- are linearly independent, then $f_+ \neq f_-$. By using f_+ or f_- in (1.2) in place of f, we have two different Fokker–Planck equations equivalent to (1.1). (As a matter of fact, we can take any linear combination of ψ_0^+ and ψ_0^-, so there are an infinite number of equivalent Fokker–Planck equations.) If ψ_0^+ and ψ_0^- are linearly dependent, then $f_+ = f_-$. In the conventional terminology of scattering theory, the cases $f_+ \neq f_-$ and $f_+ = f_-$ are referred to as 'generic' and 'exceptional', respectively. In the exceptional case, $V_-(x)$ ($= V_+(x) + \text{constant}$) is finite at both $x \to +\infty$ and $x \to -\infty$. Thus, the exceptional case corresponds to case (i) in our classification in section 5. In the generic case, on the other hand, $V_-(x)$ and $V_+(x)$ tend to $-\infty$ as $x \to +\infty$ and $x \to -\infty$, respectively. So, the generic case is included in our case (iii).

There is no particular difficulty in dealing with the exceptional case by our method; we can directly use the results of section 5 for case (i). On the contrary, special care is needed for the generic case. In the generic case, $\psi_0^-(x)$ grows linearly, and so $V_-(x)$ diverges logarithmically, as $x \to +\infty$. We can see that $\mathrm{e}^{V_-} \notin F_1^{(+)}$ since e^{V_-} behaves like $1/x^2$ as $x \to +\infty$. The criterion (6.3) for case (iii) indicates that the expansion to order k^N may not be valid for $N \geqslant 1$ if we use V_- in place of V. (The situation is the same for V_+, since $\mathrm{e}^{V_+} \notin F_1^{(-)}$.) Fortunately, we can avoid this difficulty by using both V_+ and V_-, as explained below. The idea is to use V_- for S_r, and V_+ for S_l.

The relations (2.10) hold for both V_+ and V_-, so

$$\frac{R_r^{\mathrm{S}}}{1 + R_r^{\mathrm{S}}} = S_r^+ + \frac{f_+}{2\mathrm{i}k} = S_r^- + \frac{f_-}{2\mathrm{i}k}, \qquad \frac{R_l^{\mathrm{S}}}{1 + R_l^{\mathrm{S}}} = S_l^+ - \frac{f_+}{2\mathrm{i}k} = S_l^- - \frac{f_-}{2\mathrm{i}k}. \tag{7.3}$$

Here $S_r^\pm$ and $S_l^\pm$ denote S_r and S_l with $V_\pm$ in place of V. From (2.11) and (7.3) we have

$$S(x, k) = \frac{1}{2\mathrm{i}k}[f_-(x) - f_+(x)] + S_r^-(x, k) + S_l^+(x, k). \tag{7.4}$$

The expansion of $S - 1$ takes the form of (5.9), where, instead of (5.10),

$$s_{-1} = \tfrac{1}{2}[f_-(x) - f_+(x)], \qquad s_n = a_n^{\mathrm{R}-} + a_n^{\mathrm{L}+} \quad (n \geqslant 0), \qquad \delta_N = \delta_N^{a\mathrm{R}-} + \delta_N^{b\mathrm{R}+}, \tag{7.5}$$

where $a_n^{\mathrm{R}-}$ and $a_n^{\mathrm{L}+}$ are defined by (4.6) and (4.18) with V_- and V_+, respectively, in place of V (similarly for $\delta_N^{a\mathrm{R}-}$ and $\delta_N^{b\mathrm{R}+}$). In particular, $s_0 = -\tfrac{1}{2}(\mathrm{e}^{V_-(x)} + \mathrm{e}^{V_+(x)})$. (Note that $V_1 = V_-(-\infty) = 0$ and $V_2 = V_+(+\infty) = 0$.) Substituting (7.5) into (5.12) gives

$$g_0 = \frac{-\exp\{[V_-(x) - V_+(x) - V_-(y) + V_+(y)]/4\}}{\sqrt{[f_-(x) - f_+(x)][f_-(y) - f_+(y)]}}, \tag{7.6a}$$

$$g_1 = \frac{1}{2}\left[\int_y^x (\mathrm{e}^{V_-(z)} + \mathrm{e}^{V_+(z)})\,\mathrm{d}z + \frac{\mathrm{e}^{V_-(x)} + \mathrm{e}^{V_+(x)}}{f_-(x) - f_+(x)} + \frac{\mathrm{e}^{V_-(y)} + \mathrm{e}^{V_+(y)}}{f_-(y) - f_+(y)}\right]g_0. \tag{7.6b}$$

The higher order coefficients can be calculated without any difficulty by using (7.5). Now (5.11) makes sense, and $\Delta_N = o(k^N)$ as $k \to 0$, if $V_- \in F_{N-2}^{(-)}$ and $V_+ \in F_{N-2}^{(+)}$.
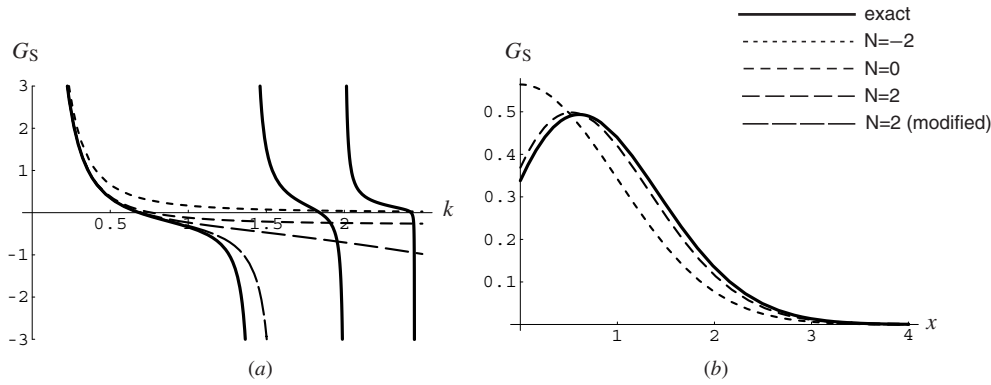
Figure 1. The Green function $G_S(x, y; k)$ for the potential $V(z) = z^2$ (example 1), (a) plotted as a function of k, with $x = 1.2$, $y = 1$; (b) plotted as a function of x, with $k = 1$, $y = 0$ (here G_S is real). In all the graphs, solid lines are the exact values. Here the dashed lines show $\sum_{m=-1}^{M}(ik)^{2m} g_{2m}$. ($N \equiv 2M = -2, 0, 2$ in (a), and $N = -2, 0$ in (b).) In (a), the curve labeled as '$N = 2$ (modified)' is the plot of (8.2).

8. Examples

To demonstrate the calculation of the expansion, let us consider some simple potentials for which the exact form of the Green function is available. (For examples 1–4, the expressions for the exact Green function can be found in section 11 of [4]. Note that $G \equiv 2ikG_S$ in [4].) In all the graphs, k is taken to be a real number ($k \geqslant 0$).

Example 1.

$$V(z) = z^2, \qquad V_S(z) = z^2 - 1.$$

The first example is a parabolic potential. The corresponding V_S is also parabolic. From (5.16), (4.11) and (4.20), we obtain the first two coefficients of (5.15) as

$$s_1(x) = \frac{\sqrt{\pi}}{2} e^{x^2}, \qquad s_3(x) = -\frac{\pi}{8} e^{x^2} \left[\int_x^\infty e^{z^2} (\operatorname{erfc} z)^2 \mathrm{d}z + \int_{-x}^\infty e^{z^2} (\operatorname{erfc} z)^2 \mathrm{d}z \right], \qquad (8.1)$$

and from (5.5) we have $q_2(x, y) = (\pi/4)(\operatorname{erfi} y - \operatorname{erfi} x)$. (Here $\operatorname{erfc} z = 1 - \operatorname{erf} z = (2/\sqrt{\pi}) \int_z^\infty e^{-w^2} \mathrm{d}w$, and $\operatorname{erfi} z = -i \operatorname{erf}(iz)$.) The expansion of the Green function has the form of (5.17), where g_{-2} and g_0 are obtained by substituting the above expressions into (5.18). The approximation to this order, $G_S \simeq (ik)^{-2} g_{-2} + g_0$, is plotted in figure 1 along with the exact value. (The exact expression of the Green function is given by equation (11.6) of [4]. Note that this G_S takes a real value when k is real.) Higher order coefficients of the expansion of G_S can be obtained in the same way. Although we omit here the expressions for s_5 and g_2, the result of the calculation up to order k^2 is also shown in figure 1(a).

The exact $G_S(k)$ has poles at $k = \pm\sqrt{2n}$ ($n = 0, 1, 2, \ldots$), corresponding to the eigenvalues of the Schrödinger operator ($k^2 = 2n$). From the information of g_0 and g_2, we can approximately reproduce the poles nearest to the origin as

$$G_S \simeq \frac{1}{(ik)^2} g_{-2} + \frac{g_0}{1 - (ik)^2 (g_2/g_0)}. \qquad (8.2)$$

This is a better approximation than $G_S \simeq (ik)^{-2} g_{-2} + g_0 + (ik)^2 g_2$ (see figure 1(a)).
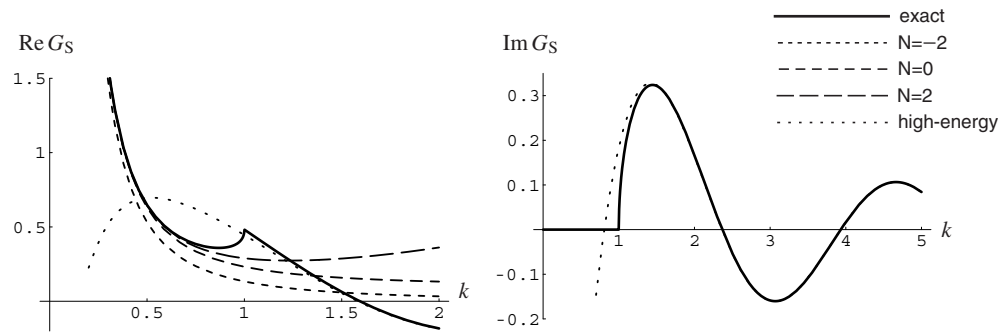
Figure 2. The real and imaginary parts of $G_S(x, y; k)$ for the potential $V(z) = 2 \log \cosh z$ (example 2), plotted as functions of k, with $x = 2$ and $y = 0$. The dashed lines show $\sum_{m=-1}^{M} (ik)^{2m} g_{2m}$. ($N \equiv 2M = -2, 0, 2$.) The dotted lines are the result of the high-energy approximation (see [4]) obtained by expanding $\log G_S$ in powers of $1/k$ to order $1/k^2$. (For $k > 1$, the curves of the high-energy approximation almost coincide with the exact curves.)

Example 2.

$$V(z) = 2 \log \cosh z, \qquad V_S(z) = 1 - 2 \mathrm{sech}^2 z.$$

In this example, $V(z)$ diverges to $+\infty$ and $V_S(z)$ tends to 1 as $z \to \pm\infty$. As in the previous example, the expansion of G_S has the form of (5.17). From (5.16), (4.11) and (4.20), we can easily calculate

$$s_1(x) = \cosh^2 x, \qquad s_3(x) = -\tfrac{1}{2} \cosh 2x \cosh^2 x, \tag{8.3}$$

and $q_2(x, y) = \frac{1}{2}(y - x) + \frac{1}{4}(\sinh 2y - \sinh 2x)$. By substituting them into (5.18), we obtain g_{-2} and g_0. The higher order coefficients can be similarly calculated. The results are shown in figure 2. (For the expression of the exact G_S, see equation (11.15) of [4].)

This G_S has branch point singularities at $k = \pm 1$, and $\mathrm{Im}\, G_S = 0$ for $|k| \leqslant 1$. The series $\sum_{m=-1}^{\infty} (ik)^{2m} g_{2m}$ is convergent for $|k| < 1$. For $|k| > 1$, we can use the high-energy expansion (discussed in [4]) to calculate G_S with very good precision (see figure 2).
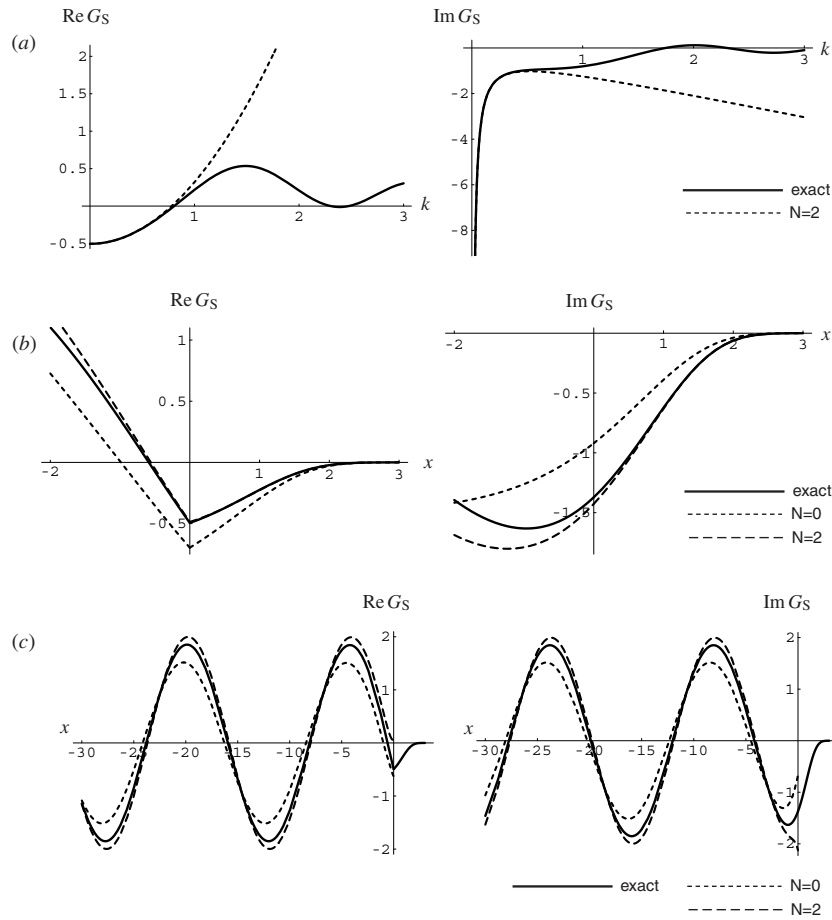
Example 3.

$$V(z) = e^z, \qquad V_S(z) = \tfrac{1}{4} e^{2z} - \tfrac{1}{2} e^z.$$

This exponential potential belongs to case (ii) of section 5. The expansions of S and G_S have the form of (5.1) and (5.3), respectively. From (5.7), (4.6) and (4.20) we have

$$s_0(z) = -\frac{1}{2} \exp(e^z), \qquad s_1(z) = -\frac{1}{2} \exp(e^z)[\mathrm{Ei}(-e^z) + 2\mathrm{Shi}(e^z)],$$

$$s_2(z) = -2 \exp(e^z) \int_{-\infty}^{z} \exp(e^w) \mathrm{Shi}(e^w)\, dw, \tag{8.4}$$

and so on, where Ei and Shi denote the exponential integral function and the hyperbolic integral function, respectively. ($\mathrm{Ei}(z) = -\int_{-z}^{\infty}(1/t)\, e^{-t}\, dt$, $\mathrm{Shi}(z) = \int_{0}^{z}(1/t) \sinh t\, dt$.) We obtain g_{-1}, g_0, etc by substituting (8.4) into (5.4), etc. The results of the calculation to order k^2 are shown in figure 3. (See equation (11.12) of [4] for the exact form of G_S.)

With fixed y and k, the function $G_S(x, y; k)$ rapidly falls off to zero as $x \to +\infty$, and oscillates as $x \to -\infty$ (recall that $G_S(x, y) = G_S(y, x)$. Since we have been assuming $y \leqslant x$, the Green function for $x < y$ is $G_S(y, x)$ with our expressions.) As can be seen from
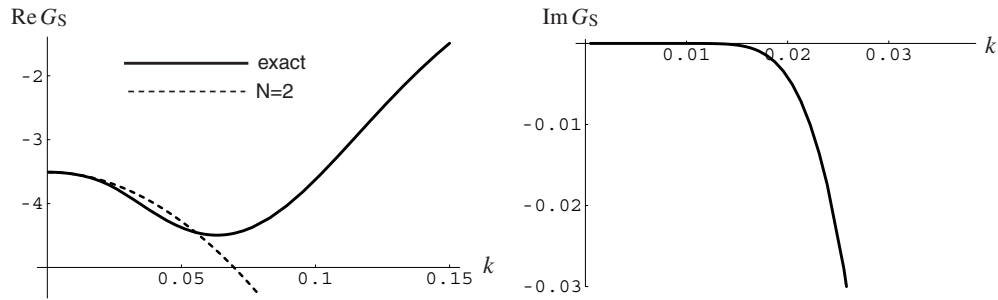
Figure 3. The real and imaginary parts of $G_S(x, y; k)$ for the potential $V(z) = e^z$ (example 3),
(*a*) plotted as functions of k, with $x = 0.5$ and $y = 0$; (*b*) plotted as functions of x, with $k = 0.4$
and $y = 0$. The dashed lines show $\sum_{n=-1}^{N}(ik)^n g_n$, where $N = 2$ in (*a*) and $N = 0, 2$ in (*b*). (Since
k is real, $N = 0$ and $N = 2$ are the same as $N = -1$ and $N = 1$, respectively, for the imaginary
part.) In (*c*), the same graphs as (*b*) ($G_S(x)$ with $k = 0.4$, $y = 0$) are drawn with a larger scope.
The dashed lines in (*c*) are the plots of (8.5) with $N = 0$ and 2. (They are plotted only for $x < 0$.)

figure 3(*b*), the truncated series $(ik)^{-1}g_{-1} + g_0 + \cdots + (ik)^N g_N$ gives a good approximation
of G_S as a function of x for $x > y$. (In figure 3(*b*), the approximation with $N = 2$ almost
coincides with the exact value for $x > y$.) However, this approximation is not effective when
$x < y$ and $y - x$ is large. To cope with the oscillatory behavior of G_S as $x \to -\infty$, it is better
to truncate the expansion of $\log G_S$ (rather than G_S itself), and then exponentiate it. Namely,

$$G_S \simeq (ik)^{-1}g_{-1} \exp[ikp_1 + (ik)^2 p_2 + \cdots + (ik)^{N+1}p_{N+1}], \qquad (8.5)$$

where $\{p_1, \ldots, p_{N+1}\}$ can be expressed in terms of $\{g_{-1}, \ldots, g_N\}$ as $p_1 = g_0/g_{-1}$, $p_2 = (g_1/g_{-1}) - \frac{1}{2}(g_0/g_{-1})^2$, etc. As shown in figure 3(*c*), this gives a good approximation in the
region where $-x$ is large.

Figure 4. The real and imaginary parts of $G_S(x, y; k)$ for the potential of example 4, plotted as functions of k, with $x = 1$ and $y = -0.5$. The dashed line is the plot of $g_0 + (ik)^2 g_2$. The falloff of $|\text{Im } G_S|$ as $k \to 0$ is faster than any power of k.

Example 4.

$$V(z) = \begin{cases} \sqrt{1-z} - 1 \\ -\sqrt{1+z} + 1, \end{cases} \qquad V_S(z) = \begin{cases} \frac{1}{16}(1-z)^{-1} + \frac{1}{8}(1-z)^{-3/2} & (z < 0) \\ \frac{1}{16}(1+z)^{-1} - \frac{1}{8}(1+z)^{-3/2} & (z > 0). \end{cases}$$

This example belongs to case (v) of section 5. Here $V(z)$ slowly diverges to $\mp\infty$ as $z \to \pm\infty$, and $V_S(z)$ tends to zero like $|z|^{-1}$. In this case, it is easier to use (5.25a) and (5.25b) directly for the calculation of g_0 and g_2. Assuming that $y < 0 < x$, we obtain

$$g_0 = -2 \exp\left(1 - \frac{\sqrt{1+x}}{2} - \frac{\sqrt{1-y}}{2}\right)(1 + \sqrt{1+x}), \tag{8.6a}$$

$$g_2 = \frac{4}{3} \exp\left(1 - \frac{\sqrt{1+x}}{2} - \frac{\sqrt{1-y}}{2}\right)[118 + 37x + 2x^2 - 3y$$
$$+ (94 + 11x - 3y)\sqrt{1+x} + 2(1-y)(1 + \sqrt{1+x})\sqrt{1-y}]. \tag{8.6b}$$

As can be seen from figure 4, equation (5.23) with (8.6) gives the correct asymptotic expansion of the Green function. (The exact Green function is given by equation (11.25) of [4] with the replacement $(x, y) \to (-y, -x)$. The $V(z)$ in this example is the same as $V(-z)$ in example 7 of [4].) The series $(ik)^{-2}g_{-2} + g_0 + (ik)^2 g_2 + \cdots$ takes a real value when k is real. Although the exact G_S is not real, the imaginary part of it approaches zero faster than any power of k as $k \to 0$ (see figure 4). Since $G_S(k)$ is essentially singular at $k = 0$, the series $\sum_{n=-1}^{\infty}(ik)^{2n}g_{2n}$ is asymptotic but divergent.

Example 5.

$$V(z) = \alpha\theta(z-1)\log z, \qquad V_S(z) = \frac{\alpha(\alpha+2)}{4}\theta(z-1)(1/z^2) - \frac{\alpha}{2}\delta(z-1),$$

where α is a positive constant, and θ denotes the Heaviside step function. ($V(z) = 0$ for $z < 1$.) Since $f(z)$ is discontinuous at $z = 1$, the Schrödinger potential contains a delta function at $z = 1$. Assume that $y < 1 < x$. From (5.8) we obtain

$$g_{-1} = x^{-\alpha/2}, \qquad g_0 = x^{-\alpha/2}\left(1 - y + \int_1^\infty \frac{1}{z^\alpha}\,\mathrm{d}z\right). \tag{8.7}$$

19
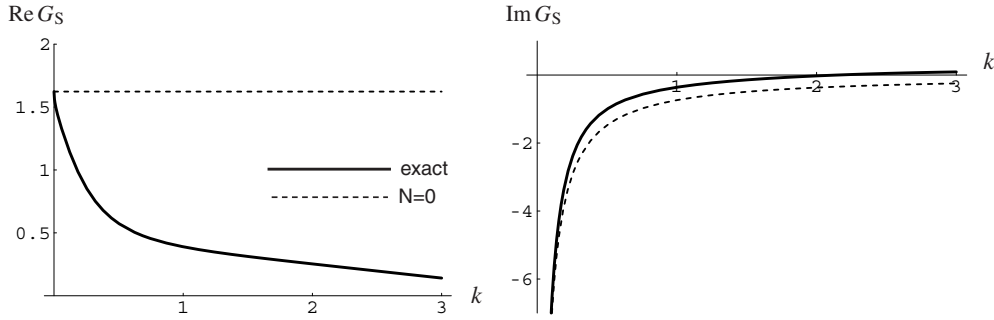
Figure 5. The real and imaginary parts of $G_S(x, y; k)$ for the potential of example 5 with $\alpha = 3/2$, plotted as functions of k, with $x = 1.5$ and $y = 0.8$. The difference between the solid and dashed lines is Δ_0. Here Δ_0 behaves like $k^{\alpha-1} = \sqrt{k}$ as $k \to 0$.

Obviously, g_0 is finite if $\alpha > 1$ $\left(e^{-V} \in F_0^{(+)}\right)$ and infinite if $\alpha \leqslant 1$ $\left(e^{-V} \notin F_0^{(+)}\right)$. The Green function for this potential can be exactly obtained as

$$G_S(x, y; k) = \frac{\sqrt{x}\left[J_\nu(kx) - e^{\nu\pi i}J_{-\nu}(kx)\right]e^{-ik(y-1)}}{k\left\{J_{\nu-1}(k) + iJ_\nu(k) + e^{\nu\pi i}\left[J_{1-\nu}(k) - iJ_{-\nu}(k)\right]\right\}}, \qquad \nu \equiv \frac{1+\alpha}{2}, \qquad (8.8)$$

where J_ν is the Bessel function. Since $J_\nu(z)$ behaves like z^ν as $z \to 0$ for non-integer ν, we can see that the asymptotic expansion of (8.8) contains a term proportional to $k^{\alpha-1}$. If $N + 1 < \alpha < N + 2$, then $\Delta_N \sim Ck^{\alpha-1}$, as explained in section 6 (see figure 5).

Example 6.

$$V_S(z) = \begin{cases} a^2 & (|z| < 1) \\ 0 & (|z| > 1), \end{cases}$$

where a is a constant. This example is to demonstrate the method discussed in section 7. The zero-energy wavefunctions $\psi_0^\pm$ for this V_S are

$$\psi_0^-(x) = \begin{cases} 1 \\ \cosh[a(x+1)] \\ C_1 + C_2 x, \end{cases} \qquad \psi_0^+(x) = \begin{cases} C_1 - C_2 x & (x < -1) \\ \cosh[a(x-1)] & (-1 < x < 1) \\ 1 & (1 < x), \end{cases} \qquad (8.9)$$

where $C_1 \equiv \cosh(2a) - a\sinh(2a)$, $C_2 \equiv a\sinh(2a)$. We obtain $V_\pm$ and $f_\pm$ from (7.2). Let us consider the case $-1 < y < x < 1$. Substituting the expressions for $V_\pm$ and $f_\pm$ into (7.6a) and (7.6b) yields

$$g_0 = -\frac{\cosh[a(x-1)]\cosh[a(y+1)]}{a\sinh(2a)}, \qquad (8.10a)$$

$$g_1 = \frac{g_0}{2a}\left(\tanh[a(x+1)] + \tanh[a(x-1)] - \tanh[a(y+1)] - \tanh[a(y-1)] + \frac{1}{\sinh(2a)}\right.$$

$$\left. \times \left\{\frac{\cosh[a(x-1)]}{\cosh[a(x+1)]} + \frac{\cosh[a(x+1)]}{\cosh[a(x-1)]} + \frac{\cosh[a(y-1)]}{\cosh[a(y+1)]} + \frac{\cosh[a(y+1)]}{\cosh[a(y-1)]}\right\}\right).$$

$$(8.10b)$$
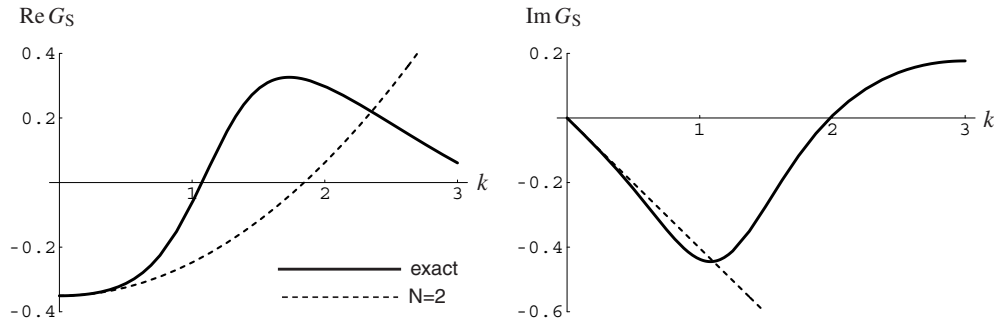
Figure 6. The real and imaginary parts of $G_S(x, y; k)$ for the potential of example 6 with $a = 1$, plotted as functions of k, with $x = 0.5$ and $y = -0.5$.

On the other hand, the exact Green function is obtained by a standard method as

$$G_S(x, y; k) = \frac{[(p - ik)\, e^{p(1-x)} + (p + ik)\, e^{-p(1-x)}][(p + ik)\, e^{-p(1+y)} + (p - ik)\, e^{p(1+y)}]}{-4p[(p^2 - k^2)\sinh(2p) - 2ipk\cosh(2p)]},$$

$$p \equiv \sqrt{a^2 - k^2}. \tag{8.11}$$

It is not difficult to check that (8.10a) and (8.10b) are the correct coefficients of the expansion. The higher order coefficients can be calculated by using (7.5) (see figure 6).

9. Summary and remarks

The asymptotic expansion of G_S in powers of k is obtained by substituting the expansion of S into (2.7). Since $S = S_r + S_l$, we can treat S_r and S_l separately. When $V(-\infty)$ is finite or $+\infty$, the coefficients of the expansion of S_r can be obtained in a simple form for arbitrary order of k (equations (4.3), (4.6), (4.10) and (4.11)). When $V(-\infty) = -\infty$, the expansion of $\left(S_r - \frac{1}{2}\right)^{-1}$, instead of S_r, takes a simple form (equations (4.13) and (4.14)). In a parallel way, corresponding expressions are obtained for S_l (equations (4.17)–(4.23)). The behavior of the remainder term, and the validity of the asymptotic expansion, can be studied by using the expressions (3.16) and (3.21). The result is given by (6.3). For the 'generic' case with $V_S(x)$ such that $V_S(+\infty) = V_S(-\infty) = 0$, we need to use a modified method explained in section 7. In this case, too, the expansion of S to arbitrary order can be obtained in a simple form (equations (7.5)).

A different method for the low-energy expansion of G_S is discussed in [17]. The method of [17], which does not use the reflection coefficients, is more direct than the method of the present paper. However, the derivation of the expansion in [17] is formal, and does not provide a way to estimate the remainder term. Practically, the methods of [17] and the present paper complement each other. It is a future problem to unify these two methods.

Appendix A. Proof of (2.10)

When $a = -\infty$, the definitions for $\bar{f}$ and $\bar{V}_S$ (equations (2.2) and (2.8)) read $\bar{f}(x) = f(x)\theta(b - x)$ and $\bar{V}_S(x) = V_S(x)\theta(b - x)$, where θ is the Heaviside step function. So,

$$\bar{f}^2(x) + \bar{f}'(x) = \bar{V}_S(x) - f(b)\delta(x - b). \tag{A.1}$$

Namely, the Schrödinger potential corresponding to $\bar{f}$ (equation (1.4) with $f \to \bar{f}$) differs from $\bar{V}_S$ by $-f(b)\delta(x - b)$. It is an elementary exercise in quantum mechanics to show that the transmission coefficient for the Schrödinger equation with the delta function potential $-f(b)\delta(x - b)$ is $T_0 \equiv 2ik/[2ik + f(b)]$, and that both the right and left reflection coefficients are equal to $R_0 \equiv -f(b)/[2ik + f(b)]$. The reflection coefficient for the Fokker–Planck equation includes the multiple reflections caused by this delta function. We can take the sum of these multiple reflections as

$$
\begin{aligned}
R_r(b, -\infty) &= R_0 + T_0^2 R_r^S(b, -\infty) \sum_{n=0}^{\infty} \left[R_0 R_r^S(b, -\infty)\right]^n = R_0 + \frac{T_0^2 R_r^S(b, -\infty)}{1 - R_0 R_r^S(b, -\infty)} \\
&= \frac{-f(b) + [2ik - f(b)]R_r^S(b, -\infty)}{2ik + f(b) + f(b)R_r^S(b, -\infty)}.
\end{aligned}
\tag{A.2}
$$

From (A.2) we obtain $R_r/(1 + R_r) = R_r^S/\left(1 + R_r^S\right) - f/(2ik)$. The second equation of (2.10) can be proved in the same way.

Appendix B. Calculation of $\bar{r}_n^a$ and $\bar{r}_n^b$

From (3.12) and (3.4) we have

$$
\begin{aligned}
\bar{r}_0^a + \xi(z, W) &= -\tanh\frac{W - V_1}{2} + \tanh\frac{W - V(z)}{2} \\
&= \frac{\sinh[V_1 - V(z)]}{\sinh[W - V(z)]}\left(\tanh\frac{W - V_1}{2} + \tanh\frac{V_1 - V(z)}{2}\right).
\end{aligned}
\tag{B.1}
$$

Substituting (B.1) into $\bar{r}_1^a = \mathcal{L}(\bar{r}_0^a + \xi)$ gives

$$
\begin{aligned}
\bar{r}_1^a(x, W) &= 2\int_{-\infty}^{x} dz\, \frac{\partial}{\partial W}\left\{\sinh[W - V(z)]\left[\bar{r}_0^a + \xi(z, W)\right]\right\} \\
&= 2\int_{-\infty}^{x} dz\, \sinh[V_1 - V(z)]\frac{\partial}{\partial W}\tanh\frac{W - V_1}{2} = \frac{e^{V_1}}{2\cosh^2\frac{W - V_1}{2}}\langle -1]_{-\infty}^{x}.
\end{aligned}
\tag{B.2}
$$

We can express $\left(\cosh^2\frac{W - V_1}{2}\right)^{-1}$ as an infinite series in powers of e^W and write

$$
\bar{r}_1^a(x, W) = 2\left[\sum_{m=1}^{\infty}(-1)^{m+1}m\, e^{m(W - V_1)}\right]e^{V_1}\langle -1]_{-\infty}^{x}.
\tag{B.3}
$$

Let us define the operators

$$
\hat{\mathcal{J}}_+^{(2)} \equiv e^W\left(1 + \frac{\partial}{\partial W}\right), \qquad \hat{\mathcal{J}}_-^{(2)} \equiv e^{-W}\left(1 - \frac{\partial}{\partial W}\right).
\tag{B.4}
$$

Then equation (3.5) can be written as

$$
\mathcal{L}g(x, W) = \sum_{\sigma=\pm 1}\int_{-\infty}^{x} e^{\sigma V(z)}\hat{\mathcal{J}}_{-\sigma}^{(2)}g(z, W)\, dz,
\tag{B.5}
$$

where $\hat{\mathcal{J}}_{-\sigma}^{(2)}$ stands for $\hat{\mathcal{J}}_-^{(2)}$ and $\hat{\mathcal{J}}_+^{(2)}$ for $\sigma = +1$ and $\sigma = -1$, respectively. Substituting (B.3) into $\bar{r}_n^a = \mathcal{L}^{n-1}\bar{r}_1^a$, and using $\int_{-\infty}^{x} dz\, e^{\sigma V(z)}\langle \cdots]_{-\infty}^{z} = \langle \ldots, \sigma]_{-\infty}^{x}$, we obtain (3.13) with

$$
D_{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}}(W) = 2e^{V_1}\hat{\mathcal{J}}_{-\sigma_{n-1}}^{(2)}\cdots\hat{\mathcal{J}}_{-\sigma_2}^{(2)}\hat{\mathcal{J}}_{-\sigma_1}^{(2)}\sum_{m=1}^{\infty}(-1)^{m+1}m\, e^{m(W - V_1)}.
\tag{B.6}
$$

Since $\hat{\mathcal{J}}_{-\sigma}^{(2)}\, \mathrm{e}^{nW} = (-\sigma)(n-\sigma)\,\mathrm{e}^{(n-\sigma)W}$, it is easy to see that

$$\hat{\mathcal{J}}_{-\sigma_{n-1}}^{(2)}\cdots\hat{\mathcal{J}}_{-\sigma_1}^{(2)}\,\mathrm{e}^{mW} = P_{\sigma_1,\dots,\sigma_{n-1}}^{(m)}\,\mathrm{e}^{(m-\Lambda)W}, \tag{B.7}$$

with $P_{\sigma_1,\dots,\sigma_{n-1}}^{(m)}$ defined by (3.15). Hence we have (3.14). Expressions of $\bar{r}_n^a$ and $\bar{\rho}_n^{a\mathrm{R}}$ without the infinite sum over m can be obtained by carrying out the calculation using the last expression of (B.2) instead of (B.3).

For the case $V(-\infty) = +\infty$, we have $\sinh[W - V(z)]\big[\bar{r}_0^b + \xi(z, W)\big] = \mathrm{e}^{W-V(z)} - 1$. Substituting this into $\bar{r}_1^b = \mathcal{L}\big(\bar{r}_0^b + \xi\big)$ (see the first line of (B.2)), we have

$$\bar{r}_1^b = 2\int_{-\infty}^{x}\mathrm{d}z\,\frac{\partial}{\partial W}(\mathrm{e}^{W-V(z)} - 1) = 2\mathrm{e}^W[-1]_{-\infty}^{x}. \tag{B.8}$$

Hence we obtain

$$\bar{r}_n^b(x, W) = 2\sum_{\{\sigma_1,\dots,\sigma_{n-1}\}}\hat{\mathcal{J}}_{-\sigma_{n-1}}^{(2)}\cdots\hat{\mathcal{J}}_{-\sigma_1}^{(2)}\,\mathrm{e}^W[-1, \sigma_1, \sigma_2, \dots, \sigma_{n-1}]_{-\infty}^{x}. \tag{B.9}$$

Equation (3.20) follows from (B.9) and (B.7).

Appendix C. Finiteness of $\bar{r}_n^a$ and $\bar{r}_n^b$

When $V(-\infty) = V_1$ is finite, we have $\mathrm{e}^{\pm V(z)} < C$ and $|\sinh[V_1 - V(z)]| < C|V(z) - V_1|$ for $-\infty \leqslant z \leqslant x$. (Here and hereafter C denotes a constant which may not necessarily be the same everywhere.) Using these inequalities in (3.2a), we find

$$\left|\langle-1, \sigma_1, \sigma_2, \dots, \sigma_{n-1}]_{-\infty}^{x}\right| < C\int_{-\infty}^{x}|V(z) - V_1|\,|z|^{n-1}\,\mathrm{d}z < \infty \tag{C.1}$$

if $V - V_1 \in F_{n-1}^{(-)}$. From (C.1) it follows that $\left|\bar{r}_n^a\right| < \infty$ if $V - V_1 \in F_{n-1}^{(-)}$.

Next, let us consider equation (3.20) for the case $V(-\infty) = +\infty$. Let us assume that $P_{\sigma_1,\dots,\sigma_{n-1}}^{(1)} \neq 0$, and let $\{p_1, p_2, \dots, p_M\}$ be a subset of $\{1, 2, \dots, n - 1\}$ such that $\sigma_{p_i} = +1$ for each i. Then there exist $\{q_1, q_2, \dots, q_M\} \in \{1, 2, \dots, n - 1\}$ such that $\sigma_{q_i} = -1$ and $q_i < p_i$ for each i. (Otherwise $P_{\sigma_1,\dots,\sigma_{n-1}}^{(1)} = 0$, as can be easily seen from (3.15).) If $-z_{p_j}$ is sufficiently large, $V(z_{q_j}) > V(z_{p_j})$. So, $\exp\big[-V(z_{q_j}) + V(z_{p_j})\big] < C$ for any $z_{q_j} < z_{p_j} < x$. We also have $\exp[-V(z)] < C$ for any $z < x$. Therefore,

$$\left|P_{\sigma_1,\dots,\sigma_{n-1}}^{(1)}[-1, \sigma_1, \sigma_2, \dots, \sigma_{n-1}]_{-\infty}^{x}\right| < C\int_{-\infty}^{x}\mathrm{e}^{-V(z)}|z|^{n-1}\,\mathrm{d}z < \infty \tag{C.2}$$

if $\mathrm{e}^{-V} \in F_{n-1}^{(-)}$. From (C.2) it is obvious that $\left|\bar{r}_n^b\right| < \infty$ if $\mathrm{e}^{-V} \in F_{n-1}^{(-)}$.

Appendix D. Proof of (4.13)

We use the rotation of coordinate axes discussed in section VI of [16]. From equations (D.1) and (D.2) of [16], we have $R_{r,\pi,-\pi/2} = -(R_{r,0,-\pi/2})^{-1}$, where $R_{r,\theta,\theta'}$ is defined by (10.2b) of [16]. When $V(-\infty) = -\infty$, the $R_r(x, -\infty)$ corresponds to $R_{r,0,-\pi/2}$. (Note that $\theta' = -\pi/2$ corresponds to $V(-\infty) = -\infty$.) Therefore,

$$-4\left(S_r - \frac{1}{2}\right) = 2\frac{1 - R_{r,0,-\pi/2}}{1 + R_{r,0,-\pi/2}}, \qquad \left(S_r - \frac{1}{2}\right)^{-1} = 2\frac{1 - R_{r,\pi,-\pi/2}}{1 + R_{r,\pi,-\pi/2}}. \tag{D.1}$$

Thus, the expression for $\left(S_r - \frac{1}{2}\right)^{-1}$ has the same form as the expression for $-4\left(S_r - \frac{1}{2}\right)$ with $\theta = \pi$ instead of $\theta = 0$. As can be seen from (6.1) and (6.3a) of [16], the rotation with angle $\theta = \pi$ amounts to changing the sign of V and k. Hence we obtain (4.13).

Appendix E. Estimation of the remainder term

Here we consider $\delta_n^{a\mathrm{R}}$ and $\delta_n^{b\mathrm{R}}$, and prove the first halves of (6.2a) and (6.2b). The conditions for $\delta_n^{a\mathrm{L}}$ and $\delta_n^{b\mathrm{L}}$ can be derived in the same way. Equations (6.2c) easily follow from (6.2b) by noting that $\delta_{n-1}^{\gamma\mathrm{R}} = o(k^{n-1})$ if $\delta_{n+1}^{\check{b}\mathrm{R}} = o(k^{n+1})$, and that $\delta_n^{\check{b}\mathrm{R}}$ is obtained from $\delta_n^{b\mathrm{R}}$ by the replacement $V \to -V$.

Let us first note that $\delta_n^{a\mathrm{R}} = o(k^n)$ and $\delta_n^{b\mathrm{R}} = o(k^n)$ hold, respectively, if $\bar{\rho}_n^a = o(k^n)$ and $\bar{\rho}_n^b = o(k^n)$ hold for $W = V(x)$. We can see this by substituting the expansion of $R_r(x, -\infty; k)$ with the remainder term (which is obtained by setting $W = V(x)$ in (3.11) or (3.18)) into the first equation of (2.5). Here we show, more generally, that $\bar{\rho}_n^a = o(k^n)$ and $\bar{\rho}_n^b = o(k^n)$ hold for any W under the conditions stated in (6.2).

For $k = 0$, the scattering coefficients can be exactly obtained. (Although equations (2.3) hold only for $k \neq 0$, we can define the scattering coefficients for $k = 0$ by taking the limit $k \to 0$.) We have [16]

$$\bar{\tau}(x, z; W; k = 0) = \operatorname{sech} \frac{W - V(z)}{2}, \tag{E.1a}$$

$$\bar{R}_l(x, z; W; k = 0) = -\bar{R}_r(x, z; W; k = 0) = \tanh \frac{W - V(z)}{2}. \tag{E.1b}$$

In this appendix, we make use of (E.1) together with the asymptotic expressions of $\bar{\tau}$ and $\bar{R}_l$ given in appendix F.

Now let us derive (6.2a). For the case $V(-\infty) = V_1$, we write (3.16) as

$$\bar{\rho}_N^a = (\mathrm{i}k)^{N+1} \sum_{\{\sigma_1,\dots,\sigma_N\}} \int_{-\infty}^{x} \mathrm{d}z\, A(z, k) \langle -1, \sigma_1, \dots, \sigma_{N-1}]_{-\infty}^z\, \mathrm{e}^{\sigma_N V(z)}, \tag{E.2}$$

where $A(z, k)$ denotes the part containing $\bar{\tau}$, $\bar{R}_l$, and the sum over m. (We omit to write the dependence on W.) Equations (E.1) yield

$$\frac{\bar{\tau}^2(x, z; k = 0)}{1 - \bar{R}_l^2(x, z; k = 0)} = 1, \qquad \frac{1 + \bar{R}_l(x, z; k = 0)}{1 - \bar{R}_l(x, z; k = 0)} = \mathrm{e}^{W - V(z)}. \tag{E.3}$$

From equations (F.23) and (F.24) of appendix F, and from the fact that C_3, C_4 and θ in these equations remain finite as $k \to 0$, we find that $\left| \bar{\tau}^2 / (1 - \bar{R}_l^2) \right| < C$, $|(1 + \bar{R}_l)/(1 - \bar{R}_l)| < C$ for $-\infty \leqslant z \leqslant x$ and $|k| < k_0$ with some k_0. (Here, too, we let C denote a constant which may not necessarily be the same at each appearance.) Using these inequalities in (3.16), we can see that $|A(z, k)| < C$. (The infinite sum in (3.16) does not cause any problems. See the comment above (3.17).) In the same way as in appendix C, it can be shown that

$$\int_{-\infty}^{x} \mathrm{d}z\, \left| \langle -1, \sigma_1, \dots, \sigma_{N-1}]_{-\infty}^z\, \mathrm{e}^{\sigma_N V(z)} \right| < \infty \tag{E.4}$$

if $V - V_1 \in F_N^{(-)}$. Since $|A(z, k)| < C$, inequality (E.4) means that the absolute value of the integrand on the right-hand side of (E.2) is dominated by a k-independent function of z which is integrable in the interval $(-\infty, x)$. Therefore, if $V - V_1 \in F_N^{(-)}$, we can interchange the order of the limit $k \to 0$ and the integral in (E.2) to obtain

$$\lim_{k \to 0} \frac{\bar{\rho}_N^a}{(\mathrm{i}k)^{N+1}} = \sum_{\{\sigma_1,\dots,\sigma_N\}} \int_{-\infty}^{x} \mathrm{d}z\, A(z, k = 0) \langle -1, \sigma_1, \dots, \sigma_{N-1}]_{-\infty}^z\, \mathrm{e}^{\sigma_N V(z)}. \tag{E.5}$$

Substituting (E.3) into (3.16), and comparing it with (3.13) and (3.14), we find that the right-hand side of (E.5) is equal to $\bar{r}_{N+1}^a$. Replacing $N \to n - 1$, we can write (E.5) as

$$\lim_{k \to 0} \frac{\bar{\rho}_{n-1}^a}{(\mathrm{i}k)^n} = \bar{r}_n^a \qquad \text{if} \quad V - V_1 \in F_{n-1}^{(-)}. \tag{E.6}$$

Since $\bar{\rho}_{n-1}^a = (ik)^n \bar{r}_n^a + \bar{\rho}_n^a$, from (E.6) it follows that $\bar{\rho}_n^a = o(k^n)$, and hence $\delta_n^{a\mathrm{R}} = o(k^n)$, as $k \to 0$ if $V - V_1 \in F_{n-1}^{(-)}$. (The proof for $n = 0$ can be done by replacing $\bar{\rho}_{n-1}^a$ with $\bar{R}_r$, and using the integral representation of $\bar{R}_r(x, -\infty)$ (equation (3.16) of [3]).)

If $V(z) - V_1 \sim \beta/|z|^\alpha (N < \alpha < N+1)$ as $z \to -\infty$, we can say more about the behavior of $\delta_N^{a\mathrm{R}}$ as $k \to 0$. In this case $\langle -1, \sigma_1, \ldots, \sigma_{N-1} \rangle_{-\infty}^z \sim C|z|^{N-\alpha}$ as $z \to -\infty$. From (F.21) and (F.22) of appendix F, we can see that the leading contribution to the integral of (E.2) has the form

$$\int_{-\infty}^x \mathrm{d}z\, A(z, k) \langle -1, \sigma_1, \ldots, \sigma_{N-1} \rangle_{-\infty}^z\, \mathrm{e}^{\sigma_N V(z)} = C \int_{-\infty}^x \mathrm{d}z\, h(\mathrm{e}^{ikz})|z|^{N-\alpha} + \cdots, \qquad (E.7)$$

where h is some function. The integral on the right-hand side is convergent if $k \neq 0$. This integral behaves like $1/k^{N+1-\alpha}$ as $k \to 0$, as can be seen by changing the integral variable from z to $Z \equiv kz$. So we have $\bar{\rho}_N \sim Ck^\alpha$, and hence $\delta_N^{a\mathrm{R}} \sim Ck^\alpha$, as $k \to 0$.

Let us proceed to (6.2b). For the case $V(-\infty) = +\infty$, we write (3.21) as

$$\bar{\rho}_N^b = (ik)^{N+1} \sum_{\{\sigma_1, \ldots, \sigma_N\}} \int_{-\infty}^x \mathrm{d}z\, B(z, k)[-1, \sigma_1, \ldots, \sigma_{N-1}]_{-\infty}^z\, \mathrm{e}^{\sigma_N V(z)}, \qquad (E.8)$$

where $B(z, k)$ is the part containing $\bar{\tau}$ and $\bar{R}_l$. Let us temporarily assume that the order of the limit $k \to 0$ and the integral in (E.8) can be interchanged. Then,

$$\lim_{k \to 0} \frac{\bar{\rho}_N^b}{(ik)^{N+1}} = \sum_{\{\sigma_1, \ldots, \sigma_N\}} \int_{-\infty}^x \mathrm{d}z\, B(z, k = 0)[-1, \sigma_1, \ldots, \sigma_{N-1}]_{-\infty}^z\, \mathrm{e}^{\sigma_N V(z)}. \qquad (E.9)$$

Substituting (E.3) into (3.21) and comparing it with (3.20), we can see that the right-hand side of (E.9) is equal to $\bar{r}_{N+1}^b$. If $\mathrm{e}^{-V} \in F_N^{(-)}$, then $\bar{r}_{N+1}^b$ is finite (see appendix C), and so (E.9) makes sense. With the replacement $N \to n - 1$, equation (E.9) reads

$$\lim_{k \to 0} \frac{\bar{\rho}_{n-1}^b}{(ik)^n} = \bar{r}_n^b \qquad \text{if} \quad \mathrm{e}^{-V} \in F_{n-1}^{(-)}. \qquad (E.10)$$

Since $\bar{\rho}_{n-1}^b = (ik)^n \bar{r}_n^b + \bar{\rho}_n^b$, it follows from (E.10) that $\bar{\rho}_n^b = o(k^n)$, and hence $\delta_n^{b\mathrm{R}} = o(k^n)$, as $k \to 0$ if $\mathrm{e}^{-V} \in F_{n-1}^{(-)}$.

Now we have only to justify the interchanging of the limit $k \to 0$ and the integral in (E.8). This is easy if $f(-\infty) \neq 0$. When $f(-\infty) \neq 0$, the behavior of $\bar{\tau}(x, z)$ as $z \to -\infty$ is given by either (F.9) or (F.16). These equations hold for $k = 0$, too, since

$$\bar{\tau}(x, z; k = 0) = 2\mathrm{e}^{W/2}\, \mathrm{e}^{-V(z)/2}[1 + o(1)] \qquad (z \to -\infty), \qquad (E.11)$$

as can be seen from (E.1a). The expressions (F.9) and (F.16) continuously approach (E.11) as $k \to 0$. Since the quantities $\eta(z, k)$ and $\theta(z, k)$ in (F.9) and (F.16) are $o(|z|)$ as $z \to -\infty$, we have, for $|k| < k_0$ with some k_0,

$$|\bar{\tau}^2(x, z; k)| < C \exp[-V(z) + C'z], \qquad (E.12)$$

where C' is a constant which can be chosen arbitrarily small. Considering the behavior of $\bar{R}_l$ given either by (F.10) or (F.17), we see that

$$\left| \frac{1}{1 - \bar{R}_l^2} \left(\frac{1 + \bar{R}_l}{1 - \bar{R}_l} \right)^{1-\Lambda} \right| = \frac{1}{|1 - \bar{R}_l|^2} \left| \frac{1 + \bar{R}_l}{1 - \bar{R}_l} \right|^{-\Lambda} < C, \qquad (E.13)$$

since $\Lambda \leqslant 0$. From (E.13) and (3.21), we find

$$|B(z, k)| < C\bar{\tau}^2(x, z; k)\, \mathrm{e}^{(1-\Lambda)V(z)}. \qquad (E.14)$$

25

In the same way as in appendix C, it can be shown that[6]

$$\left| [-1, \sigma_1, \ldots, \sigma_{N-1}]_{-\infty}^z \, e^{\sigma_N V(z)} \, e^{(1-\Lambda)V(z)} \right| < C \, e^{V(z)} \int_{-\infty}^z e^{-V(w)} |w|^{N-1} \mathrm{d}w < C|z|^N. \quad (E.15)$$

These inequalities hold as long as $e^{-V} \in F_{N-1}^{(-)}$. From (E.12), (E.14) and (E.15) we have

$$\left| B(z,k)[-1, \sigma_1, \ldots, \sigma_{N-1}]_{-\infty}^z \, e^{\sigma_N V(z)} \right| < C|z|^N \exp[-V(z) + C'z]. \quad (E.16)$$

Since $f(-\infty) \neq 0$, in this case $V(z)$ tends to $+\infty$ linearly or faster as $z \to -\infty$. Therefore, the right-hand side of (E.16) is integrable in the interval $(-\infty, x)$. The absolute value of the integrand of (E.8) is thus dominated by a k-independent integrable function, and this justifies the interchanging of the limit and the integral.

When $V(-\infty) = +\infty$ and $f(-\infty) = 0$ (i.e., when $V(z)$ grows slower than linearly), we cannot find a k-independent integrable function of z that dominates $|\bar{\tau}^2(x, z; k)|$ as in (E.12). In this case, the behavior of $\bar{\tau}(x, z)$ as $z \to -\infty$ is given by (F.23). However small k may be, $\bar{\tau}(x, z; k)$ is considerably different from $\bar{\tau}(x, z; 0)$ when $-z$ is large, since (F.23) is not compatible with (E.11). The crossover of the two different behavior takes place in the region where $|k| \simeq |f(z)|$. (This can be known by studying the small-k expansion of $\bar{\tau}$.) Let us define z_k by $|k| = |f(z_k)|$. (Such z_k is uniquely determined when k is sufficiently small, since we are assuming that $f(z)$ is asymptotically monotone.) Roughly speaking, $\bar{\tau}(x, z; k) \simeq \bar{\tau}(x, z; 0)$ for $z > z_k$ when k is sufficiently small. For $z < z_k$, we need to use (F.23). The factor C_3 in (F.23) is of the order of $e^{-V(z_k)/2}$, since $\bar{\tau}(x, z_k; k)$ is of the same order as $\bar{\tau}(x, z_k; 0) \simeq C \, e^{-V(z_k)/2}$. We can write

$$\bar{\tau}(x, z; k) = C \, e^{-V(z_k)/2} \exp[-\mathrm{i}k(z - z_k) + \mathrm{i}\theta(z_k, z, k)][1 + o(1)], \quad (E.17)$$

where θ is defined by (F.20). We divide the integral in (E.8) as $\int_{-\infty}^x = \int_{-\infty}^{z_k} + \int_{z_k}^x$. The part $\int_{z_k}^x$ can be treated in the same way as in the case $f(-\infty) \neq 0$. An inequality analogous to (E.12) holds for $z \geqslant z_k$, and it can be shown that $\lim_{k \to 0} \int_{z_k}^x = \int_{-\infty}^x \lim_{k \to 0}$. Let us study the part $\int_{-\infty}^{z_k}$. Equation (E.17) gives[7]

$$\int_{-\infty}^{z_k} \bar{\tau}^2(x, z; k)|z|^N \mathrm{d}z \simeq C \, e^{-V(z_k)}/k^{N+1} \quad (E.18)$$

if we neglect the $\theta(z_k, z, k)$ and the $o(1)$ part of (E.17). (By using (F.18)–(F.21), it can be shown that the contributions from θ and the $o(1)$ part in (E.17) are indeed negligible in the limit $k \to 0$.) Using (E.15), (E.17) and (F.24), we can estimate the part $\int_{-\infty}^{z_k}$ of (E.8). This is essentially the same as (E.18), and we have

$$\left| \int_{-\infty}^{z_k} B(z,k)[-1, \sigma_1, \ldots, \sigma_{N-1}]_{-\infty}^z \, e^{\sigma_N V(z)} \mathrm{d}z \right| < C \, e^{-V(z_k)}/k^{N+1}. \quad (E.19)$$

(Also see (E.7) and the explanation below it.) If $V(z) \sim (N + 1) \log|z|$ as $z \to -\infty$, then $|z_k| \sim C/|k|$ as $k \to 0$. Since $e^{-V(z_k)} \sim C|z_k|^{-(N+1)} \sim C|k|^{N+1}$, the right-hand side of (E.19) approaches a finite value as $k \to 0$. If $e^{-V} \in F_N^{(-)}$, then $V(z)$ grows faster than $(N + 1) \log|z|$ as $z \to -\infty$, and so the right-hand side of (E.19) vanishes in the limit $k \to 0$. Therefore, the part $\int_{-\infty}^{z_k}$ of (E.8) is negligible as $k \to 0$ if $e^{-V} \in F_N^{(-)}$, and, since $\lim_{k \to 0} \int_{z_k}^x = \int_{-\infty}^x \lim_{k \to 0}$, this means that $\lim_{k \to 0}$ and $\int_{-\infty}^x$ can be interchanged.

If $V(z) \sim \alpha \log|z|$ ($N < \alpha < N + 1$) as $z \to \infty$, the inequalities in (E.15) can be replaced by '$\sim$', and (E.19) gives $\sum \int_{-\infty}^{z_k} B(z, k) \cdots \mathrm{d}z \sim Ck^{\alpha-N-1}$ ($k \to 0$). We can

---

[6] In appendix F of [3], it is assumed that the quantity on the left-hand side of (E.15) tends to a finite value as $z \to -\infty$, but this is wrong.

[7] When Im $k = 0$, it is necessary to replace k by $k + \mathrm{i}\epsilon$ and let $\epsilon \downarrow 0$ after evaluating the integral.

also see that $\sum \int_{z_k}^x B(z,k) \cdots \mathrm{d}z \sim C \int_{z_k}^x |z|^{-\alpha+N} \, \mathrm{d}z \sim C|z_k|^{-\alpha+N+1} \sim Ck^{\alpha-N-1}$ as $k \to 0$. (This is obtained by substituting (E.3) into the integrand.) Therefore, from (E.8) we obtain $\bar{\rho}_N^b \sim Ck^{\alpha}$, and hence $\delta_N^{b\mathrm{R}} \sim Ck^{\alpha}$, as $k \to 0$.

Appendix F. Asymptotic behavior of $\bar{\tau}(x, z)$ and $\bar{R}_l(x, z)$ as $z \to -\infty$

In this appendix, we study the asymptotic forms of $\bar{\tau}(x, z; W; k)$ and $\bar{R}_l(x, z; W; k)$ as $z \to -\infty$ for $\mathrm{Im}\, k \geqslant 0, k \neq 0$. Details of the derivation are omitted, but let us only mention that equations (F.5), (F.14), (F.15), (F.21) and (F.22), which are the basic expressions, are all derived by using equations (3.5)–(3.8) of [16]. We need to consider the three cases, (1) $f(-\infty) = \pm\infty$, (2) $f(-\infty) = c \neq 0$ and (3) $f(-\infty) = 0$.

(1) $f(-\infty) = \pm\infty$.
Let us consider the Schrödinger equations

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi^{\pm}(z) + [\mp f'(z) + f^2(z)]\psi^{\pm}(z) = k^2\psi^{\pm}(z). \tag{F.1}$$

The equation for ψ^- is identical with (1.1), and the equation for ψ^+ is the Schrödinger equation corresponding to the inverted Fokker–Planck potential $-V$. We set

$$\psi^{\pm}(z) \equiv \exp\left[\frac{1}{2}V(z) + \int^z \tilde{p}^+(w, k) \, \mathrm{d}w\right], \tag{F.2}$$

and substitute into (F.1). This gives the nonlinear differential equations

$$\frac{\partial}{\partial z}\tilde{p}^{\pm}(z, k) \mp 2f(z)\tilde{p}^{\pm}(z, k) + [\tilde{p}^{\pm}(z, k)]^2 = -k^2. \tag{F.3}$$

It is easy to see that these equations have solutions satisfying the asymptotic conditions

$$\tilde{p}^{\pm}(z, k) = \pm\frac{k^2}{2f(z)}[1 + o(1)] \qquad \text{as} \quad z \to -\infty. \tag{F.4}$$

We can express the scattering coefficients in terms of these solutions $\tilde{p}^{\pm}$ as

$$\tau(x, z; k) = 1/\alpha^+, \qquad R_l(x, z; k) = -\beta^-/\alpha^+, \tag{F.5}$$

$$\begin{aligned}
\alpha^+ &\equiv \frac{1}{2[k^2 + \tilde{p}^+(x, k)\tilde{p}^-(x, k)]}\{[k + \mathrm{i}\tilde{p}^-(x, k)][k - \mathrm{i}\tilde{p}^+(z, k)]\,\mathrm{e}^{(1/2)[V(z)-V(x)]-\eta^+} \\
&\quad + [k + \mathrm{i}\tilde{p}^+(x, k)][k - \mathrm{i}\tilde{p}^-(z, k)]\,\mathrm{e}^{(1/2)[V(x)-V(z)]-\eta^-}\}, \\
\beta^- &\equiv \frac{1}{2[k^2 + \tilde{p}^+(x, k)\tilde{p}^-(x, k)]}\{[k + \mathrm{i}\tilde{p}^-(x, k)][k + \mathrm{i}\tilde{p}^+(z, k)]\,\mathrm{e}^{(1/2)[V(z)-V(x)]-\eta^+} \\
&\quad - [k + \mathrm{i}\tilde{p}^+(x, k)][k + \mathrm{i}\tilde{p}^-(z, k)]\,\mathrm{e}^{(1/2)[V(x)-V(z)]-\eta^-}\}
\end{aligned} \tag{F.6}$$

where $\eta^{\pm} \equiv \int_z^x \tilde{p}^{\pm}(w, k) \, \mathrm{d}w$. (The symbols α^+ and β^- are in accordance with the notation used in section III of [16].) If $f(-\infty) = +\infty$, then $V(-\infty) = +\infty$, and (F.5) gives the asymptotic behavior of $\tau(x, z; k)$ as $z \to -\infty$

$$\tau(x, z; k) = C(x, k)\exp\left[-\frac{1}{2}V(z) + \eta(z, k)\right][1 + o(1)], \tag{F.7}$$

$$\eta(z, k) \equiv \int_z^{z_0} \tilde{p}^+(w, k) \, \mathrm{d}w = \frac{k^2}{2}\int_z^{z_0} \frac{1}{f(w)} \, \mathrm{d}w[1 + o(1)], \tag{F.8}$$

where z_0 is a constant. The generalized transmission coefficient $\bar{\tau}(x, z; W; k)$ is the transmission coefficient for the potential that has a jump at the right end point x (see [16]). So it is obvious that $\bar{\tau}$ has the same asymptotic form as (F.7). The expression for $\bar{R}_l$ is also the same as that for R_l. We have

$$\bar{\tau}(x, z; W; k) = C_1(x, W, k) \exp\left[-\frac{1}{2}V(z) + \eta(z, k)\right][1 + o(1)], \qquad \text{(F.9)}$$

$$\bar{R}_l(x, z; W; k) = -1 - \frac{ik}{f(z)}[1 + o(1)]. \qquad \text{(F.10)}$$

From (F.8) we can see that $\eta(z) = o(|z|)$ as $z \to -\infty$. If $1/f \in F_0^{(-)}$, then $\eta(-\infty, k)$ is finite, and so we may let $\eta = 0$ in (F.9) by including $e^{\eta(-\infty, k)}$ in C_1. Since $\lim_{k \to 0} \tilde{p}^{\pm} = 0$, we have $C(k = 0) = e^{V(x)/2}$, $C_1(k = 0) = 2e^{W/2}$ and $\eta(z, k = 0) = 0$.

The results for the case $f(-\infty) = -\infty$ can be obtained in the same way. The asymptotic expressions for $\bar{\tau}$ and $\bar{R}_l$ are obtained by replacing V with $-V$ in (F.9), and by changing the sign of the first term on the right-hand side of (F.10).

(2) $f(-\infty) = c$.
We consider the second-order differential equation

$$-\frac{d^2}{dz^2}\psi(z) + f^2(z)\psi(z) + \frac{f'(z)}{f(z)}\left[\frac{d}{dz}\psi(z) - ik\psi(z)\right] = k^2\psi(z). \qquad \text{(F.11)}$$

(This is the equation satisfied by the quantities $\alpha^+(x, z)$ and $\beta^+(x, z)$ defined in section III of [16], as functions of z with fixed x.) We set $\psi(z) \equiv \exp\left[\int^z q(w, k)\,dw\right]$, and substitute into (F.11). This yields the differential equation

$$\frac{\partial}{\partial z}q(z, k) - \frac{f'(z)}{f(z)}q(z, k) + q^2(z, k) = f^2(z) - k^2 - \frac{f'(z)}{f(z)}ik. \qquad \text{(F.12)}$$

Suppose that $k^2 \neq c^2$. (Since we are interested in the region of small k, we need not consider the case $k^2 = c^2$.) Then equation (F.12) has a solution that tends to $-\sqrt{c^2 - k^2}$ as $z \to -\infty$. Let us define, in terms of this solution $q(z, k)$,

$$p(z, k) \equiv q(z, k) + \sqrt{c^2 - k^2}, \qquad s(z, k) \equiv [ik - q(z, k)]/f(z),$$
$$\gamma(x, z, k) \equiv \frac{1}{2}\int_z^x [p(w, k) + p(w, -k)]\,dw,$$
$$\theta(x, z, k) \equiv \frac{1}{2i}\int_z^x [p(w, k) - p(w, -k)]\,dw, \qquad \text{(F.13)}$$
$$\Theta(x, z, k) \equiv i\sqrt{c^2 - k^2}(x - z) + \theta(x, z, k).$$

Then we have

$$\tau(x, z; k) = \frac{1 - s(x, k)s(x, -k)}{1 - s(x, k)s(z, -k)\,e^{2i\Theta(x,z,k)}}\,e^{i\Theta(x,z,k)+\gamma(x,z,k)}, \qquad \text{(F.14)}$$

$$R_l(x, z; k) = \frac{-s(z, k) + s(x, k)\,e^{i\Theta(x,z,k)}}{1 - s(x, k)s(z, -k)\,e^{2i\Theta(x,z,k)}}. \qquad \text{(F.15)}$$

We can show from (F.12) that $p(z, k) = B(k)[f(z) - c][1 + o(1)]$ as $z \to -\infty$, where $B(k)$ is some z-independent quantity. Since $p(z, k)$ is $o(1)$ as $z \to -\infty$, obviously both $\gamma(x, z, k)$ and $\theta(x, z, k)$ are $o(|z|)$ as $z \to -\infty$. It is not difficult to show that, in fact, $\gamma(x, -\infty, k)$

is finite. Note also that $\lim_{z \to -\infty} s(z, k) = (ik + \sqrt{c^2 - k^2})/c$. As long as $\sqrt{c^2 - k^2}$ has a nonzero real part, $e^{i\Theta(x,z,k)}$ vanishes in the limit $z \to -\infty$, and we obtain

$$\bar{\tau}(x, z; W; k) = C_2(x, W, k) \exp[\sqrt{c^2 - k^2} z + i\theta(z, k)][1 + o(1)], \qquad \text{(F.16)}$$

$$\bar{R}_l(x, z; W; k) = -\frac{1}{c}(ik + \sqrt{c^2 - k^2}) + o(1), \qquad \text{(F.17)}$$

where $\theta(z, k) \equiv \theta(z_0, z, k)$ with a fixed constant z_0. This $\theta(z, k)$ is $o(|z|)$ as $z \to -\infty$. The expressions for τ and R_l are of the same forms, with W replaced by $V(x)$.

(3) $f(-\infty) = 0$.
Substituting $\psi(z) \equiv \exp\left[ikz + \int^z p(w, k)\,dw\right]$ into (F.11) yields the differential equation

$$\frac{\partial}{\partial z} p(z, k) + \left[2ik - \frac{f'(z)}{f(z)}\right] p(z, k) + p^2(z, k) = f^2(z). \qquad \text{(F.18)}$$

This equation has a solution that has the asymptotic form, as $z \to -\infty$,

$$p(z, k) = \frac{1}{2ik} f^2(z)[1 + o(1)] \qquad \text{if} \quad \lim_{z \to -\infty} \frac{f'(z)}{f(z)} = 0, \qquad \text{(F.19a)}$$

$$p(z, k) = \frac{1}{b + 2ik} f^2(z)[1 + o(1)] \qquad \text{if} \quad \lim_{z \to -\infty} \frac{f'(z)}{f(z)} = b, \qquad \text{(F.19b)}$$

$$p(z, k) = -\frac{1}{2} f(z)[V(z) - V(-\infty)][1 + o(1)] \qquad \text{if} \quad \lim_{z \to -\infty} \frac{f'(z)}{f(z)} = \infty. \qquad \text{(F.19c)}$$

(Recall that $f'(z)$ is monotone for sufficiently large $|z|$, by our assumption. The right-hand side of (F.19b) is replaced by $zf^2(z)[1 + o(1)]$ when $b + 2ik = 0$.) Let p stand for the solution specified by (F.19). Just like (F.13), we define

$$s(z, k) \equiv -p(z, k)/f(z), \qquad \gamma(x, z, k) \equiv \frac{1}{2} \int_z^x [p(w, k) + p(w, -k)]\,dw,$$

$$\theta(x, z, k) \equiv \frac{1}{2i} \int_z^x [p(w, k) - p(w, -k)]\,dw. \qquad \text{(F.20)}$$

Then we have

$$\tau(x, z; k) = \frac{1 - s(x, k)s(x, -k)}{1 - s(x, k)s(z, -k)\,e^{2i[k(x-z)+\theta(x,z,k)]}} e^{ik(x-z)+i\theta(x,z,k)+\gamma(x,z,k)}, \qquad \text{(F.21)}$$

$$R_l(x, z; k) = \frac{-s(z, k) + s(x, k)\,e^{2i[k(x-z)+\theta(x,z,k)]}}{1 - s(x, k)s(z, -k)\,e^{2i[k(x-z)+\theta(x,z,k)]}}. \qquad \text{(F.22)}$$

From (F.19) it follows that $s(z, \pm k) = o(1)$ as $z \to -\infty$. It is not difficult to see that $\gamma(x, -\infty, k)$ is finite. So (F.21) and (F.21) yield the asymptotic forms as $z \to -\infty$

$$\bar{\tau}(x, z; W; k) = C_3(x, W, k) \exp[-ikz + i\theta(z, k)][1 + o(1)], \qquad \text{(F.23)}$$

$$\bar{R}_l(x, z; W; k) = C_4(x, W, k) \exp[-2ikz + 2i\theta(z, k)] + o(1), \qquad \text{(F.24)}$$

where $\theta(z, k) \equiv \theta(z_0, z, k)$ with a constant z_0. This $\theta(z, k)$ is $o(|z|)$ as $z \to -\infty$. If $f^2 \in F_0^{(-)}$, then $\theta(-\infty, k)$ is finite, and we may let $\theta = 0$ in these expressions by including $e^{i\theta(-\infty,k)}$ in C_3. The expressions for τ and R_l have the same forms as (F.23) and (F.24), with W replaced by $V(x)$.

Appendix G. The existence of $R_r(x, -\infty; k)$

We defined the reflection coefficients for semi-infinite intervals as $R_r(x, -\infty; k) \equiv \lim_{y \to -\infty} R_r(x, y; k)$. (For $\mathrm{Im}\, k = 0$, the limit $\epsilon \downarrow 0$ of $k + i\epsilon$ is implied when necessary, as in (2.4).) Here we show that such a limit exists for $\mathrm{Im}\, k \geqslant 0$ (the existence of $R_l(\infty, y; k)$ can be shown in the same way). We use the integral representation [16, 18]

$$R_r(x, y; k) = \int_y^x f(z) \tau^2(x, z; k)\, dz \tag{G.1}$$

and the asymptotic form of $\tau(x, z; k)$ as $z \to -\infty$ given in appendix F. For $k = 0$, we have exactly $R_r(x, y; k = 0) = \tanh\{[V(y) - V(x)]/2\}$ (see (E.1)). We can let $y \to -\infty$ in this expression (see (3.7)). In the following, we assume that $k \neq 0$.

First, we consider the case $f(-\infty) = +\infty$. Substituting (F.7) into (G.1) gives

$$R_r(x, y; k) = C^2(x, k) \int_y^x f(z)[1 + h(z)]\, e^{-V(z) + 2\eta(z,k)}\, dz, \tag{G.2}$$

where $h(z) = o(1)$ as $z \to -\infty$. Since $f(z) = -(1/2)(d/dz)V(z)$,

$$\int_y^x f(z)\, e^{-V(z) + 2\eta(z)}\, dz = -\frac{1}{2}\, e^{-V(y) + 2\eta(y)} + \int_y^x \eta'(z)\, e^{-V(z) + 2\eta(z)}\, dz + C, \tag{G.3}$$

where C is independent of y. In this case, $V(z)$ tends to $+\infty$ faster than $|z|$. From appendix F we know that $\eta(z) = o(|z|)$ and $\eta'(z) = o(1)$ as $z \to -\infty$. So the first term on the right-hand side of (G.3) vanishes, and the second term is convergent, as $y \to -\infty$. It is obvious that the part including $h(z)$ in (G.2) is also convergent as $y \to -\infty$, since the integrand decays even faster. Therefore, the limit $y \to -\infty$ of $R_r(x, y; k)$ exists, irrespective of whether $\mathrm{Im}\, k > 0$ or $\mathrm{Im}\, k = 0$. The argument is the same for the case $f(-\infty) = -\infty$.

Next, let us consider the case $f(-\infty) = 0$. We write (F.21) as

$$\tau(x, z; k) = \frac{B\, e^{-i[kz - \theta(z)] + \gamma(z)}}{1 - u(z)\, e^{-2i[kz - \theta(z)]}} \tag{G.4}$$

explicitly writing only the dependence on z, with $u(z) \equiv s(z, -k)s(x, k)\, e^{2ikx}$ and $B \equiv [1 - s(x, k)s(x, -k)]\, e^{ikx}$. If $\mathrm{Im}\, k > 0$, then $\tau(x, z; k)$ falls off exponentially as $z \to -\infty$, and it is easy to show that (G.1) has the limit $y \to -\infty$. So, let us assume that $\mathrm{Im}\, k = 0$. If $\mathrm{Im}\, k = 0$, the quantities $\theta(z)$ and $\gamma(z)$ (defined by (F.20)) are real-vaued functions of z, since $p(z, -k) = [p(z, k)]^*$. By our assumption, $f(z)$ and $f'(z)$ are monotone for sufficiently large $|z|$. From (F.18) and (F.19) it follows that $p(z, \pm k)$, and hence $\gamma(z), \theta(z), \theta'(z), \mathrm{Re}\, u(z), \mathrm{Im}\, u(z)$, are all monotone for sufficiently large $|z|$. As can be seen from appendix F,

$$\begin{aligned} \theta(z) = o(|z|), &\qquad \theta'(z) = o(1), \\ \gamma(z) = O(1), &\qquad u(z) = o(1) \qquad \text{as} \quad z \to -\infty. \end{aligned} \tag{G.5}$$

We divide the integral of (G.1) in two parts as $\int_y^x = \int_{y_0}^x + \int_y^{y_0}$ with some y_0, and substitute (G.4) into the second part. Then

$$\int_y^{y_0} f(z) \tau^2(x, z; k)\, dz = B^2 \int_y^{y_0} f(z)\, e^{2\gamma(z)} \left(\sum_{n=0}^{\infty} n u^n(z)\, e^{-2(n+1)i[kz - \theta(z)]} \right) dz. \tag{G.6}$$

Here we have expressed τ^2 as a power series in terms of u. This infinite series is convergent if $-y_0$ is chosen to be sufficiently large. Let us define

$$I_n \equiv n B^2 \int_y^{y_0} f(z)\, e^{2\gamma(z)} u^n(z)\, e^{-2(n+1)i[kz - \theta(z)]}\, dz. \tag{G.7}$$

Setting $Z \equiv z - [\theta(z)/k]$, we can write

$$I_n = n B^2 \int_{z=y}^{z=y_0} A(Z) \, e^{-2(n+1)ikZ} \, dZ, \qquad A(Z) \equiv \frac{f(z) \, e^{2\gamma(z)} u^n(z)}{1 - [\theta'(z)/k]}. \tag{G.8}$$

Since $f(z), \gamma(z), u(z), \theta'(z)$ are all asymptotically monotone, there exists a number w such that $A(Z)$ is monotone for $z < w$, and from (G.5) we see that $A(Z) \to 0$ as $z \to -\infty$. Let us take $y_0 < w$. Then, using the second mean value theorem of integral calculus, we find

$$|\text{Re } I_n| < 2 \left| B^2 \frac{f(y_0) \, e^{2\gamma(y_0)}}{k - \theta'(y_0)} \right| |u(y_0)|^n, \qquad |\text{Im } I_n| < 2 \left| B^2 \frac{f(y_0) \, e^{2\gamma(y_0)}}{k - \theta'(y_0)} \right| |u(y_0)|^n. \tag{G.9}$$

When y is finite, term-by-term integration is permissible on the right-hand side of (G.6). The infinite series $\sum_{n=0}^{\infty} I_n$ is convergent and is equal to the left-hand side of (G.6). Taking the sum of (G.9) over n, we obtain

$$\left| \text{Re} \int_y^{y_0} f(z) \tau^2(x, z; k) \, dz \right| < 2 \left| B^2 \frac{f(y_0) \, e^{2\gamma(y_0)}}{k - \theta'(y_0)} \right| \frac{1}{1 - |u(y_0)|}, \tag{G.10}$$

and similarly for the imaginary part. (We are taking $-y_0$ to be sufficiently large so that $|u(y_0)| < 1$ and also $k > \theta'(y_0)$.) The right-hand side of (G.10) is independent of y, and vanishes as $y_0 \to -\infty$. Therefore, the limit $y \to -\infty$ of (G.6) exists. Hence it is obvious that $\lim_{y \to -\infty} R_r(x, y; k)$ exists.

The case $f(-\infty) = c \neq 0$ can be studied in the same way, using (F.14) instead of (F.21). If $\text{Re} \sqrt{c^2 - k^2} \neq 0$, then $\tau(x, y; k)$ falls off exponentially as $y \to -\infty$, and it is easy to show the existence of the limit $y \to -\infty$ of (G.1). If k is real and $k^2 > c^2$, the right-hand side of (G.1) oscillates as $y \to -\infty$ and does not converge. In order to make the integral have a definite value, it is necessary to assume that k has an infinitesimal imaginary part $i\epsilon$. Then $\lim_{\epsilon \downarrow 0} \lim_{y \to -\infty}$ of (G.1) exits, as can be seen from the fact that the following limit exists:

$$\lim_{\epsilon \downarrow 0} \lim_{Y \to -\infty} \int_Y^X \frac{e^{-2iKZ + \epsilon Z}}{(1 - a \, e^{-2iKZ + \epsilon Z})^2} \, dZ = \frac{-e^{-2iKX}}{2iK(1 - a \, e^{-2iKX})}. \tag{G.11}$$

(Apart from a constant factor, the integrand on the left-hand side of (G.11) is the asymptotic form of $f(z) \tau^2(x, z; k)$ as $z \to -\infty$, where $K \equiv \sqrt{k^2 - c^2}$ and $Z \equiv z - [\theta(z)/K]$. The remaining part of $f(z) \tau^2(x, z; k)$ vanishes as $z \to -\infty$, and its integral converges more rapidly.) In this paper, however, we need not be concerned with the case $k^2 > c^2$, since we are studying the small-k expansion. The case $k^2 = c^2$ is not discussed here, since we do not need this case either.

References

[1] Risken H 1984 *The Fokker–Planck Equation* (Berlin: Springer)
[2] Miyazawa T 2006 *J. Phys. A: Math. Gen.* **39** 10871
[3] Miyazawa T 2006 *J. Phys. A: Math. Gen.* **39** 7015
   Miyazawa T 2006 *J. Phys. A: Math. Gen.* **39** 15059 (corrigendum)
[4] Miyazawa T 2007 *J. Phys. A: Math. Theor.* **40** 8683
[5] Newton R G 1966 *Scattering Theory of Waves and Particles* (New York: McGraw-Hill)
[6] Deift P and Trubowitz E 1979 *Commun. Pure Appl. math.* **32** 121
[7] Chadan K and Sabatier P C 1989 *Inverse Problems in Quantum Scattering Theory* 2nd edn (New York: Springer)
[8] Yafaev D R 1982 *Comm. Math. Phys.* **85** 177
[9] Bollé D, Gesztesy F and Wilk S F J 1985 *J. Operator Theory* **13** 3
[10] Newton R G 1986 *J. Math. Phys.* **27** 2720
[11] Klaus M 1988 *J. Math. Phys.* **29** 148
[12] Klaus M 1988 *Inverse Problems* **4** 505
[13] Aktosun T and Klaus M 1999 *J. Math. Phys.* **40** 3701

[14] Aktosun T 2000 *J. Math. Phys.* **41** 4262
[15] Aktosun T and Klaus M 2001 *Inverse Problems* **17** 619
[16] Miyazawa T 1998 *J. Math. Phys.* **39** 2035
[17] Miyazawa T 1999 *J. Math. Phys.* **40** 838
     Miyazawa T 2000 *J. Math. Phys.* **41** 6861
[18] Miyazawa T 1995 *J. Math. Phys.* **36** 5643

Corrigendum

Low-energy asymptotic expansion of the Green function for one-dimensional Fokker-Planck and Schrödinger equations
Toru Miyazawa 2008 *J. Phys. A: Math. Theor.* **41** 315304

There is an error in the definition of the transmission coefficient (equations (2.3)). The transmission coefficient τ in (2.3a) and (2.3b) should be multiplied by $e^{[V(b)-V(a)]/2}$ and $e^{-[V(b)-V(a)]/2}$, respectively. Without this correction, the functions ϕ_1 and ϕ_2 defined by (2.3) are solutions of the Schrödinger equation, not the Fokker–Planck equation.

The same error occurs in the corresponding equations in the previous series of papers (equations (1.6) of [1], equations (3.2) of [2], and equations (2.10) of [3]). The functions defined by these equations should be interpreted as solutions of the Schrödinger equation instead of the Fokker–Planck equation. Otherwise, there should be a factor $e^{[V(x_2)-V(x_1)]/2}$ (for $x < x_1$) or $e^{-[V(x_2)-V(x_1)]/2}$ (for $x > x_2$) in front of $\tau(x_2, x_1; k)$. This error does not affect any of the results of this paper or these previous papers.

References

[1] Miyazawa T 2006 *J. Phys. A: Math. Gen.* **39** 7015
[2] Miyazawa T 2006 *J. Phys. A: Math. Gen.* **39** 10871
[3] Miyazawa T 2007 *J. Phys. A: Math. Theor.* **40** 8683